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Effective completeness theorems for modal logic

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Abstract

We initiate the study of computable model theory of modal logic, by proving effective completeness theorems for a variety of first-order modal logics. We formulate a natural definition of a decidable Kripke model, and show how to construct such a decidable Kripke model of a given decidable theory. Our construction is inspired by the effective Henkin construction for classical logic. The Henkin construction, however, depends in an essential way on the Deduction Theorem. In its usual form the Deduction Theorem fails for modal logic. In our construction, the Deduction Theorem is replaced by a result about objects called finite Kripke diagrams. We argue that this result can be viewed as an analogue of the Deduction Theorem for modal logic.

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1. Introduction

The formalization of computability and the subsequent development of computability (or recursion) theory in the middle part of the last century made it possible to investigate the effective content of various mathematical structures and constructions. This area of study has flourished since its inception in the 1960s and 1970s, culminating in the recent publication of the *Handbook of Recursive Mathematics* [2]. We refer the reader to the Introduction of that text for a comprehensive history and overview of the subject.

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One of the primary areas of Recursive Mathematics is computable model theory. Here we refer to the subject as *classical* computable model theory, since it has focused on the model theory of classical (first-order) logic. The first result of classical computable model theory is the effective completeness theorem: every decidable theory has a decidable model. The proof of this theorem consists of an effective version of Henkin's construction. We present this proof in detail in Section 2.1. As we shall see, Henkin's construction depends in an essential way on the Deduction Theorem.

More recently, research has begun into the computable model theory of non-classical logics. Ishihara, Khoussainov, and Nerode have proved an effective completeness theorem for intuitionistic logic [9]. Their construction of a decidable intuitionistic Kripke model adapted the effective Henkin construction for classical first-order logic to the context of intuitionistic Kripke models. To do so, they took advantage of the fact that the Deduction Theorem holds for intutionistic logic.

This work continues the development of the computable model theory of non-classical logics by initiating the study of the computable model theory of modal logics. In Section 3, we prove an effective completeness theorem for a particular modal logic: first-order constant domain **K**. In Sections 5 and 6 we generalize this result to numerous other first-order modal logics: **K**, **T**, **K5**, **S4**, and **S5**, with any of three comain domain assumptions (constant domains, monotonically increasing domains, varying domains).

Our constructions of decidable Kripke models for these modal logics draw on the ideas behind Henkin's construction. But, of course, the Deduction Theorem (in its usual form) fails for modal logic. We develop a technique for effectively constructing Kripke models which is analogous to Henkin's method, using structures which we call finite Kripke diagrams. The use of the Deduction Theorem in the classical Henkin construction is replaced by a result about finite Kripke diagrams, which we call the Testing Lemma. We argue in Section 4 that this Testing Lemma can be viewed as an analogue of the Deduction Theorem for modal logic.

We begin in the following section with some preliminaries: a recap of the effective Henkin construction for classical first-order logic, and the necessary definitions for modal logic.

2. Preliminaries

2.1. Henkin's construction

The effective completeness theorem for classical first-order logic is proved by simply noticing that the standard Henkin construction can be carried out effectively for a decidable theory T. As we mentioned, it is the starting point of the subject of classical computable model theory; indeed, it is the first theorem in the first chapter of the *Handbook of Recursive Mathematics* [7, pp. 18–19]. We repeat the proof here, so that we can compare and contrast it with the our effective construction for modal logic.

Let $\mathscr{A}$ be a model for a classical first-order language L, with domain A. Then L^A is the expansion of L obtained by adding a new constant symbol c_a for each $a \in A$. We may assume that $\mathscr{A}$ is a model for the language L^A, in which each such c_a names a. Recall that the complete diagram of $\mathscr{A}$ is the set of all sentences of L^A which are true in $\mathscr{A}$.

Also recall that a model $\mathscr{A}$ is *decidable* if its complete diagram is computable. A *decidable theory* is a computable set of sentences closed under logical consequence. We will assume throughout that every theory is consistent.

Construction 2.1. Given a decidable theory T, the construction will produce a model $\mathscr{A}$. Fix a computable enumeration $c_0, c_1, c_2, \ldots$ of an infinite set of new constants C (i.e., $L \cap C = \emptyset$), and fix a computable enumeration $\phi_0, \phi_1, \phi_2, \ldots$ of all sentences in the extended language $L \cup C$.

We will effectively build a complete diagram Δ of a model of T in stages, by satisfying each *eth completeness requirement* at some finite stage: at each stage, we will add either ϕ_e or $\neg\phi_e$ to Δ. At each finite stage we will have a finite approximation (denoted by $\{\psi_0, \psi_1, \ldots, \psi_{n-1}\}$ below) to Δ. The Deduction Theorem is used to effectively decide—based on the decidability of T—whether to add ϕ_e or $\neg\phi_e$ at a given stage. The model defined by Δ is decidable since Δ is built as a computable set.

More precisely, define a sequence of formulas $\psi_0, \psi_1, \psi_2, \ldots$ as follows:

- Stage 0: $\psi_0 = \top$
- Stage $n = 2e + 1$ (satisfy the eth completeness requirement): Let $\delta_n = \psi_0 \wedge \psi_1 \wedge \cdots \wedge \psi_{n-1}$. Effectively check whether

  $$T \models \delta_n \rightarrow \neg\phi_e.$$

  If so, let $\psi_n = \neg\phi_e$. Otherwise ($T \not\models \delta_n \rightarrow \neg\phi_n$), let $\psi_n = \phi_e$.
- Stage $n = 2e + 2$ (satisfy Henkin witness requirement): If $\psi_{n-1} = \exists x\theta(x)$, let $\psi_n = \theta(c_i)$, where c_i is the least element of C which does not occur in δ_{n-1}.

Now we define a model $\mathscr{A}$ from Δ. The domain of $\mathscr{A}$ is C (assume for now that L does not include equality). The interpretation $R^{\mathscr{A}}$ of each atomic relation symbol R is determined by the contents of Δ. For each n-place relation symbol R and n-tuple $\bar{c}$ from C,

$$\bar{c} \in R^{\mathscr{A}} \quad \Leftrightarrow \quad R(\bar{c}) \in \Delta.$$

The definition of $R^{\mathscr{A}}$ serves as the base case of the induction in the proof of the following result.

Lemma 2.2 (Truth Lemma). *For every sentence ϕ of $L \cup C$,*

$$\mathscr{A} \models \phi \quad \Leftrightarrow \quad \phi \in \Delta.$$

By the construction, Δ is computable and contains T. Thus, the Truth Lemma establishes that $\mathscr{A}$ is a decidable model of T, completing the proof of the effective completeness theorem for classical first-order logic:

Theorem 2.3. *A decidable theory T in a classical first-order language L has a decidable model.*

Note that the Deduction Theorem is essential to this construction. It is used in the key step, to effectively decide how to satisfy the eth completeness requirement: we add $\neg\phi_e$ at stage $n = 2e + 1$ iff $T \models \delta_n \to \neg\phi_e$. This is because are committed to adding $\neg\phi_e$ to Δ at stage $n = 2e + 1$ if the finite approximation to Δ constructed so far (δ_n) semantically entails $\neg\phi_e$ over T, i.e., if every model of $T \cup \{\delta_n\}$ is also a model of $\neg\phi_e$. This is the LHS of the Deduction Theorem:

$$T \cup \{\delta_n\} \models \neg\phi_e \quad \Leftrightarrow \quad T \models \delta_n \to \neg\phi_e.$$

The Deduction Theorem allows us to effectively decide whether the LHS holds is true, because we can using the decidability of T to effectively check whether the RHS is true. Thus, the Deduction Theorem provides a syntactic characterization (whether $T \models \delta_n \to \neg\phi_e$) of the relevant semantic condition (whether every model of $T \cup \{\delta_n\}$ is also a model of $\neg\phi_e$).

Also note that we fulfill the Henkin witness requirements so that the Truth Lemma is satisfied. If we put an existential sentence $\exists x\theta(x) \in \Delta$, then we need to make sure $\mathscr{A} \models \exists x\theta(x)$. To that end we also put $\theta(c_i) \in \Delta$, for some $c_i \in C$.

Both these aspects of the Henkin construction, the essential use of the Deduction Theorem and the Henkin witnesses, will be important as we turn our attention to modal logics and Kripke models. One, we will need to work around the Deduction Theorem, since it fails for modal logic. And two, we will use the idea behind the Henkin witness requirement to deal with the modalities (specifically $\diamond$) of modal logic.

2.2. *First-order modal logic*

We will prove an effective completeness theorem for first-order constant domain modal logic, with the basic $\diamond$ and $\Box$ modalities. First we define the syntax and semantics of the logic, and introduce computability into the semantics.

2.2.1. *Syntax*

We define a first-order modal language L over a given set of relation symbols, constant symbols, and variables. The *terms* of L are the constant symbols and variables. The *atomic formulas* of L are all expressions of the form $P(t_1, \ldots, t_n)$, where P is an n-place relation symbol and the t_i are terms. The set of *formulas* of L is defined as follows:

- each atomic formula is a formula;
- if ϕ is a formula, then so is $\neg\phi$;
- if ϕ, ψ are formulas, then so is $\phi \wedge \psi$;
- if ϕ is a formula and x is a variable, then so is $\exists x\phi$;
- if ϕ is formula, then so are $\diamond\phi$, $\Box\phi$.

2.2.2. Semantics

Definition 2.4. A (*constant domain*) *Kripke model* $M = (W, R, D, I)$ for a language L consists of

- a non-empty set of possible worlds W;
- a binary possibility relation $R \subseteq W \times W$;
- a non-empty domain D;
- an interpretation I such that
  - $I(w)(P) \subseteq D^n$ for every n-place relation symbol P of L;
  - $I(c) \in D$ for every constant symbol c of L.

We may view a Kripke model as a collection of classical models, one for each possible world $w \in W$, tied together by the possibility relation R. In this section we restrict our attention to constant domain Kripke models, in which the classical models have the same domain D.[1]

As with predicate logic, we can expand a Kripke model M by adding new constants for each of the elements in the domain. Given a modal language L and a Kripke model M for L, we extend L to a language L^M by adding a constant symbol c_a for each $a \in D$. Then M is expanded to a model for L^M by setting $I(c_a) = a$ for each $a \in D$.

We define the conditions under which a sentence ϕ is true at a world w in a model M (notation: $(M, w) \models \phi$) by induction on the structure of ϕ:

- $(M, w) \models P(c_0, \ldots, c_n)$ iff $(I(c_0), \ldots, I(c_n)) \in I(w)(P)$;
- $(M, w) \models \neg\phi$ iff $(M, w) \not\models \phi$;
- $(M, w) \models \phi \wedge \psi$ iff $(M, w) \models \phi$ and $(M, w) \models \psi$;
- $(M, w) \models \exists x \phi(x)$ iff $(M, w) \models \phi(c_a)$ for some $a \in D$;
- $(M, w) \models \diamond\phi$ iff $(M, w') \models \phi$ for some w' such that $(w, w') \in R$;
- $(M, w) \models \Box\phi$ iff $(M, w') \models \phi$ for every w' such that $(w, w') \in R$.

We have defined what it means for a sentence to be true at a world w in a model M. A sentence ϕ is true in a model M (notation: $M \models \phi$) iff $(M, w) \models \phi$ for every $w \in W$. For a set of sentences T, M is a model of T (notation: $M \models T$) iff $M \models \phi$ for every $\phi \in T$.

Finally, we have a notion of logical consequence, which will be essential as we turn our attention to theories. A sentence ϕ is a logical consequence of a set of sentences T (notation: $T \models \phi$) iff for every model M such that $M \models T$, it is also the case that $M \models \phi$.

Lemma 2.5. *$T \not\models \phi$ iff there exists a model $M = (W, \ldots)$ and a world $w \in W$ such that $M \models T$ but $(M, w) \models \neg\phi$.*

[1] Note that we have made no restriction on the possibility relation R; it can be any binary relation whatsoever. Thus, the logic we have defined is constant domain **K**. In Sections 5 and 6 we will investigate two standard generalizations of constant domain models (varying domain models and monotonic domain Kripke models), and also look at the logics defined by placing restrictions on the possibility relations (**T**, **K4**, **S4**, etc.).

This follows from the preceding definitions.

Finally, as in predicate logic, a *theory* in a modal language L is a set of sentences of L closed under logical consequence.

2.3. Decidable Kripke models

Now we introduce computability into the semantics of modal logic, by generalizing the notion of a decidable model to Kripke models.

Definition 2.6. A Kripke model $M=(W,R,D,I)$ for the language L is *decidable* if the sets W,D and the relation R are each computable, and truth at a world is decidable, i.e., the relation

$$\{(w,\phi):(M,w)\models\phi\}$$

is computable, where ϕ ranges over the sentences of L^M.

A classical first-order model is decidable if the truth of each sentence in the model is computable. In a Kripke model, truth of a sentence is relative to a given world in the model. So for a Kripke model to be decidable, the truth of a given sentence in a given world must be computable. We also require the possibility relation to be computable.

A *decidable theory* is a computable set of sentences of a modal language L which is closed under logical consequence, i.e., a theory T is decidable iff $T=\{\phi : T\models\phi\}$ is a computable set.

Our goal is to prove an effective completeness theorem for modal logic, by constructing a decidable Kripke model of a given decidable theory.

2.4. The canonical model

Our construction of a decidable Kripke model will combine aspects of two well-known constructions: the (effective) Henkin construction for first-order classical first-order logic, as presented in Construction 2.1; and the canonical model construction for modal logics, which we sketch in this section.

The canonical model construction is often used to prove completeness theorems for various modal logics. [2] There are complications in carrying out the canonical model construction for first-order modal logics (for a detailed discussion, see [6]), but for propositional modal logic the construction is relatively straightforward. So we present the canonical model construction in the context of propositional modal logic, though we use it as an inspiration for our construction in the context of first-order modal logic.

Instead of taking a detour into the details of propositional modal logic, let us just say that propositional modal logic is defined much as first-order modal logic was in Section 2.2, but with a set of atomic propositions in place of the atomic relation symbols and atomic formulas. Then a propositional Kripke model takes the form

[2] See, for example, [1,3,4,8] for detailed presentations of the canonical model construction for various modal logics.

$M=(W,R,I)$, where W and R are as before, but now $I(w)(P)\in\{T,F\}$ for each atomic proposition P.

Given a theory T in a propositional modal language, the canonical model for T is defined as follows:

- the set of possible worlds W is the set of all maximal consistent sets of formulas which contain T;
- the possibility relation R is defined by the following syntactic relation between the maximal consistent sets:

$$(v,w)\in R \quad\Leftrightarrow\quad \{\phi : \Box\phi\in v\}\subseteq w.$$

- the interpretation I is determined by the contents of w:

$$I(w)(P)=T \quad\Leftrightarrow\quad P\in w.$$

Note that the interpretation I of the atomic propositions in the canonical model is defined by the contents of w, in exactly the same way that the interpretation I of the atomic relation symbols is defined from the contents of the set Δ in the classical first-order Henkin construction.

In fact, we can view the canonical model construction as an adaptation of Henkin's construction to modal logic and Kripke models. In particular, if we carry out Henkin's construction for classical propositional logic, the resulting set Δ is a maximal consistent set. Whereas this single maximal consistent set suffices to build a classical model, we need multiple maximal consistent sets to build a Kripke model with multiple worlds. For the canonical model, we take *all* maximal consistent sets. Then the interpretation I and the possibility relation R are defined so that a Truth Lemma goes through, just as it did in Henkin's construction.

Lemma 2.7 (Truth Lemma). *For every sentence ϕ and every world w in the canonical model M,*

$$(M,w)\models\phi \quad\Leftrightarrow\quad \phi\in w.$$

Clearly the canonical model is a model of T, since we took as the possible worlds the maximal consistent sets containing T. Hence, the canonical model for T is indeed a model of T.

Now consider the effective content of the canonical model construction. We saw that the classical Henkin construction can be easily "effectivized" for a decidable classical first-order theory T. But for decidable modal theory T, the canonical model construction cannot simply be effectivized to produce a decidable model, because of the "top down" manner in which it is defined. The possible worlds are defined as *all* maximal consistent sets, which does not give an effective enumeration of the set of possible worlds. Nor does it give an effective account of the contents of each maximal consistent set, which is necessary in order to make truth in the model decidable (via the Truth Lemma). Finally, the possibility relation is put in after the possible worlds have been defined, according to a syntactic relation between the maximal consistent sets which appears to be undecidable.

We will fix this by building a Kripke model of T which is similar to the canonical model; but we will construct the model from the "bottom up." We will construct just the maximal consistent sets we need, effectively enumerating the sets and their contents in the course of the construction. Moreover, the possibility relation between these sets will be present as we construct them. All this will make the resulting Kripke model decidable.

3. An effective completeness theorem

In this section we will prove our main result (Theorem 3.11): every decidable theory of modal logic has a decidable Kripke model. We motivate our proof by first sketching how we propose to carry out an effective "bottom-up" version of the canonical model construction.

3.1. Proposed construction

Suppose T is a decidable theory in a first-order modal language L. We begin by fixing a computable set of new constants C, and an effective enumeration $\phi_0, \phi_1, \phi_2, \ldots$ of the sentences in the extended language $L \cup C$.

So far the setup is exactly as for the classical first-order Henkin construction. The new constants C will serve as the elements of the model, and the construction will consist primarily of satisfying completeness requirements with respect to the enumeration of sentences.

We will build a Kripke model with multiple worlds, but since we want to build it from the "bottom up," we will begin with only a single world. Call it w_0. As in the canonical model, each possible world will be a set of sentences. But now we want to effectively enumerate the contents of each possible world, so initially w_0 is empty. We then start satisfying completeness requirements at w_0, and adding Henkin witnesses from C for existential sentences.

Recall that the Henkin witnesses are needed in order to satisfy the Truth Lemma. We will want our Kripke model M to satisfy a Truth Lemma as well. Thus, if we put a sentence $\exists x\psi(x) \in w_0$, then we want to make sure that $(M, w_0) \models \exists x\psi(x)$. To that end, we put $\psi(c) \in w_0$ for some "new" $c \in C$.

Now let us turn to the important new aspect of modal logic: the modalities. The semantics of the $\diamond$ modality are very similar to those of the existential quantifier. As is often remarked, the $\diamond$ modality acts as a sort of existential quantifier on possible worlds. So we will need to satisfy Henkin witness requirements for $\diamond$-formulas—but with respect to possible worlds. We need to do this to satisfy the semantics of the $\diamond$ modality: if we put a sentence $\diamond\psi \in w_0$, we need to make sure that $(M, w_0) \models \diamond \psi$. To that end, we create a new world w_1 such that $(w_0, w_1) \in R$ and $\psi \in w_1$. We call w_1 a $\diamond$-*witness*, since it witnesses the semantics of $\diamond\psi \in w_0$.

Once we have created a new world w_1, we need to start building a complete diagram at w_1 also. If in doing so we add a $\diamond$-formula $\diamond\theta$ to w_0 or w_1, we create a new world w_3, accessible from w_0 or w_1, respectively, with $\theta \in w_3$.

$$\begin{tikzcd}
 & & w_2 & \\
 & w_1 \arrow[ur] \arrow[r] \arrow[dr] & w_4 & \\
w_0 \arrow[ur] \arrow[dr] & & w_5 & w_7 \\
 & w_3 \arrow[r] & w_6 \arrow[ur] \arrow[dr] & \\
 & & & w_8
\end{tikzcd}$$

Fig. 1. Sketch of a finite Kripke diagram.

Thus, at each stage of the construction we will have a finite graph, consisting of finitely many "existing worlds" $w_0, w_1, \ldots, w_p$ and a finite possibility relation R. We will need to build a complete diagram for each such existing world, so we will need to satisfy each completeness requirement at each existing world over the course of the construction. Moreover, whenever we add a $\diamond$-formula to an existing world, we immediately create a new world as a successor to that existing world, as a $\diamond$-witness; that is how the possibility relation R is built up. In summary, at each stage we will have a finite approximation to a Kripke model.

We may depict such an approximation by a finite (directed) graph, as in Fig. 1. Here, the existing worlds $w_0, w_1, \ldots, w_8$ are finite sets of sentences—namely, the sentences we have added to satisfy completeness requirements at those worlds, and subsequently from satisfying Henkin and $\diamond$-witness requirements. The arrows represent the relation R, built up by creating $\diamond$-witnesses.

Thus, w_1 and w_3 were created as $\diamond$-witnesses for sentences added to w_0, meaning there are sentences $\diamond\psi, \diamond\theta \in w_0$ such that $\psi \in w_1$ and $\theta \in w_3$. Similarly, w_2, w_4 and w_5 were created as $\diamond$-witnesses for sentences added to w_3; w_6 was created as a $\diamond$-witness for a sentence added to w_3; w_7 and w_8 were created as $\diamond$-witnesses for sentences added to w_6.

3.2. Finite Kripke diagrams

According to this construction sketch, each finite stage will produce a certain kind of finite approximation to a Kripke model. The classical Henkin construction also produced a finite approximation to (the complete diagram of) a classical first-order model at each stage, consisting of a finite set of sentences ($\{\psi_0, \psi_1, \ldots, \psi_{n-1}\}$ in the notation of Construction 2.1). We refer to those sets as "finite diagrams," since they are finite approximations to the complete diagram of the model.

In the modal case, the finite approximations to the Kripke model will be objects of the kind described above: a finite number of finite sets of sentences, together with a binary possibility relation on those sets. These will be the primary objects in the construction, which we call finite Kripke diagrams.

Definition 3.1. A *finite Kripke diagram* (FKD) $D=(W=\{w_0,\dots,w_n\},R)$ for a modal language L consists of a finite set of *existing worlds* W, where each $w_i\in W$ is a finite set of sentences of L; and a binary relation $R\subseteq W\times W$.

As in Fig. 1, we can view a FKD $D=(W,R)$ as a finite graph with nodes W and edge relation R, with a finite number of sentences (the contents of w_i) associated with each node $w_i\in W$. Also note that in the construction we have proposed, R is extended only when we add new worlds as immediate R-successors to existing worlds. Thus, the graph (W,R) will be a tree for each FKD we construct, as in Fig. 1. We call these *tree FKDs*; unless otherwise specified, each FKD we discuss will be a tree FKD.

We plan to extend a given FKD $D=(W,R)$ during a given stage in the construction by satisfying the eth completeness requirement at some existing world $w_i\in W$. But we must take care to do that in such a way that the contents of the various existing worlds do not contradict each other. For example, consider the FKD shown in Fig. 2.

Clearly we want to avoid constructing a FKD such as this one, since it cannot lead to a model M which satisfies the Truth Lemma: there is no such model M such that $(w_0,w_1),(w_1,w_2)\in R$ and in which $(M,w_0)\models\Box\Box\phi$, $(M,w_1)\models\Box(\neg\phi\vee\neg\psi)$, and $(M,w_2)\models\psi$.

This example illustrates that the contents of different existing worlds in W can "interact" via the possibility relation R and the modalities of the language. Thus, when we go to extend the contents of a given existing world w_i, we have to somehow take into account the contents of the other worlds in W as well.

What is it about the FKD in Fig. 2 that we want to avoid? In the classical Henkin construction, we extended the finite diagrams in such a way as to maintain "semantical consistency." The construction ensured that there is always *some* model of T in which all of the sentences in the finite diagram were simultaneously true.

We must do something similar in the modal case. We must be sure that there is a Kripke model which "witnesses" each FKD we build, in the following sense.

Definition 3.2. A Kripke model $M=(W_M,R_M,\dots)$ *witnesses* a FKD $D=(W_D,R_D)$ via f if M is a model for the language of D and $f:W_D\to W_M$ is a map such that

- if $(w_i,w_j)\in R_D$, then $(f(w_i),f(w_j))\in R_M$;
- if $\phi\in w_i$, then $(M,f(w_i))\models\phi$.

Definition 3.3. A FKD D is *T-consistent* if there exists a Kripke model M of T which witnesses D.

$$w_0=\{\Box\Box\phi\}\longrightarrow w_1=\{\Box(\neg\phi\vee\neg\psi)\}\longrightarrow w_2=\{\psi\}$$

Fig. 2. An FKD to avoid.

Thus, the FKD in Fig. 2 is not T-consistent (for any theory T), since no Kripke model witnesses it.

3.3. The key step

Suppose we have a T-consistent FKD D_n at the beginning of stage n of the proposed construction. Our main task in stage n will be to satisfy some eth completeness requirement at some existing world w_i of D_n, i.e., we must add either ϕ_e or $\neg\phi_e$ to w_i. Moreover, we must *effectively* decide which to add, and we must choose so that the new "extended diagram" is also T-consistent.

Because we will repeatedly refer to such extended diagrams, we introduce a notation for them. For a FKD $D=(W=\{w_0,\dots,w_n\},R)$ and a sentence ϕ, let $D+\{\phi\in w_i\}$ denote the FKD

$$(W=\{w_0,\dots,w_{i-1},w_i\cup\{\phi\},w_{i+1},\dots,w_n\},R)$$

i.e., the result of extending D by adding ϕ to w_i.

Thus, at stage n we must effectively determine that either $D_n+\{\phi_e\in w_i\}$ is T-consistent, or that $D_n+\{\neg\phi_e\in w_i\}$ is T-consistent. Notice that since D_n is T-consistent, we know that at least one of them is T-consistent. We reason as follows. Since D_n is T-consistent, there is a Kripke model M of T which witnesses D_n via f. Now either $(M,f(w_i))\models\phi_e$ or $(M,f(w_i))\models\neg\phi_e$. Hence, M also witnesses either $D_n+\{\phi_e\in w_i\}$ or $D_n+\{\neg\phi_e\in w_i\}$, also via f.

So the key to the construction is to be able to effectively decide whether $D_n+\{\phi_e\in w_i\}$ is T-consistent. If it is, then we can set $D_{n+1}=D_n+\{\phi_e\in w_i\}$. On the other hand, if $D_n+\{\phi_e\in w_i\}$ is *not* T-consistent, then $D_n+\{\neg\phi_e\in w_i\}$ must be T-consistent, and so we can set $D_{n+1}=D_n+\{\phi_e\in w_i\}$.

Now recall the analogous step of the classical Henkin construction. There, the Deduction Theorem allowed us to make this decision effectively. The finite diagram at stage n is just $\{\psi_0,\dots,\psi_{n-1}\}$, and the extended diagram is simply $\{\psi_0,\dots,\psi_{n-1}\}\cup\{\phi_e\}$. Note that $\{\psi_0,\dots,\psi_{n-1}\}\cup\{\phi_e\}$ is T-consistent (i.e., there is classical first-order model $\mathscr{A}$ such that $\mathscr{A}\models\psi_0\wedge\cdots\wedge\psi_{n-1}\wedge\phi_e$) iff it is not the case that every model of $T\cup\{\psi_0,\dots,\psi_{n-1}\}$ is also a model of $\neg\phi_e$, i.e., $T\cup\{\psi_0,\dots,\psi_{n-1}\}\not\models\neg\phi_e$. By the Deduction Theorem, this is equivalent to

$$T\not\models\delta_n\rightarrow\neg\phi_e,$$

where $\delta_n=\psi_0\wedge\cdots\wedge\psi_{n-1}$. Since T is decidable, it followed that the decision about ϕ_e is decidable. Thus, the Deduction Theorem for classical first-order logic is used to decide semantic consistency via a syntactic criterion: $\{\psi_0,\dots,\psi_{n-1}\}\cup\{\phi_e\}$ is (semantically) T-consistent iff $T\cup\{\delta_n\}\not\models\neg\phi_e$ iff $T\not\models\delta_n\rightarrow\neg\phi_e$, where the last equivalency follows from the Deduction Theorem.

Now suppose we rephrase the syntactic criterion, by noting that the implication $\delta_n\rightarrow\neg\phi_e$ is equivalent to $\neg(\delta_n\wedge\phi_e)$. Hence, in the classical case, we can T-consistently to add ϕ_e to the given finite diagram δ_n iff

$$T\not\models\neg(\delta_n\wedge\phi_e).$$

Instead of asking if the finite diagram δ_n implies $\neg\phi_e$ (over T), we ask if $\delta_n \wedge \phi_e$, which represents the extended diagram $\{\psi_0, \ldots, \psi_{n-1}\} \cup \{\phi_e\}$, is "syntactically T-consistent."[3]

This is the point of view we will generalize to handle the modal case. We will add ϕ_e to the current FKD—but now at a particular world w_i—and then check whether the resulting finite diagram is T-consistent.

To make this decision effectively, we need to syntactically "represent" a finite Kripke diagram within the modal language L. That will allow us to use the decidability of the theory T to (syntactically) decide its (semantic) consistency. In the case of classical logic, a finite diagram is represented by simply the conjunction of its contents ($\delta_n \wedge \phi_e$ above); the situation for modal logic is more complicated.

3.4. The representing formula and the testing lemma

We showed in the previous section that we will be able to carry out the key step of the construction if we can syntactically represent a given FKD in such a way that we can use the decidability of T to effectively decide whether the FKD is T-consistent. We shall define a formula Ψ^D which represents a tree FKD D in this sense: D is T-consistent iff $T \not\models \neg\Psi^D$. We will use the tree structure of D to define Ψ^D.

Definition 3.4. Suppose we have a tree FKD $D = (W = \{w_0, \ldots, w_n\}, R)$ with root node w_0. We associate with each existing world w_i a formula Ψ_i, defined by induction as follows:

- If w_i is a leaf node, then $\Psi_i = \bigwedge\{\phi \mid \phi \in w_i\}$.
- If w_i is a not a leaf, then $\Psi_i = \bigwedge(\{\phi \mid \phi \in w_i\} \cup \{\diamond\Psi_j \mid (w_i, w_j) \in R\})$.

Finally, let $\Psi^D = \Psi_0$, the formula associated with the root node. We call Ψ^D the *representing formula* of D.

We have defined the formulas Ψ_i recursively, with the leaves of the tree as the base case. We can view the formula Ψ_i as encoding the part of the FKD "in front of" w_i, i.e., at w_i and at all worlds reachable from w_i via R. As we work our way back from the leaves to the root, the result is the formula Ψ^D, which carries all the information of the entire FKD. This idea is made precise by Lemma 3.5, which shows that Ψ^D is precisely what we need to test the T-consistency of a FKD D.

First, let us present an example. Consider the FKD D presented in Fig. 3. In this example, $\Psi_2 = \phi_2 \wedge \psi_2 \wedge \theta_2$, $\Psi_4 = \phi_4 \wedge \psi_4$, and $\Psi_5 = \theta_5$; so

$$\begin{aligned}\Psi_1 &= \phi_1 \wedge \psi_1 \wedge \diamond\Psi_2 \wedge \diamond\Psi_4 \wedge \diamond\Psi_5\\ &= \phi_1 \wedge \psi_1 \wedge \diamond(\phi_2 \wedge \psi_2 \wedge \theta_2) \wedge \diamond(\phi_4 \wedge \psi_4) \wedge \diamond\theta_5.\end{aligned}$$

[3] In fact, we might argue that this is a more natural view, since $\{\psi_0, \ldots, \psi_{n-1}\} \cup \{\phi_e\}$ is T-consistent iff there is classical first-order model $\mathscr{A}$ of T such that $\mathscr{A} \models \delta_n \wedge \phi_e$; i.e., iff $T \not\models \neg(\delta_n \wedge \phi_e)$.

$$
\begin{tikzcd}
 & & w_2 = \{\phi_2, \psi_2, \theta_2\} & \\
 & w_1 = \{\phi_1, \psi_1\} \arrow[ur] \arrow[r] \arrow[dr] & w_4 = \{\phi_4, \psi_4\} & \\
w_0 = \{\phi_0\} \arrow[ur] \arrow[dr] & & w_5 = \{\theta_5\} & w_7 = \{\phi_7, \psi_7, \theta_7\} \\
 & w_3 = \{\phi_3\} \arrow[r] & w_6 = \{\phi_6, \theta_6\} \arrow[ur] \arrow[dr] & \\
 & & & w_8 = \{\psi_8\}
\end{tikzcd}
$$

Fig. 3. FKD D for representing formula example.

Also, $\Psi_7 = \phi_7 \wedge \psi_7 \wedge \theta_7$ and $\Psi_8 = \psi_8$, so

$$
\begin{aligned}
\Psi_6 &= \psi_6 \wedge \theta_6 \wedge \diamond\Psi_7 \wedge \diamond\Psi_8 \\
&= \psi_6 \wedge \theta_6 \wedge \diamond(\phi_7 \wedge \psi_7 \wedge \theta_7) \wedge \diamond(\psi_8).
\end{aligned}
$$

and so

$$
\begin{aligned}
\Psi_3 &= \phi_3 \wedge \diamond\Psi_6 \\
&= \phi_3 \wedge \diamond(\psi_6 \wedge \theta_6 \wedge \diamond(\phi_7 \wedge \psi_7 \wedge \theta_7) \wedge \diamond\psi_8).
\end{aligned}
$$

Finally

$$
\begin{aligned}
\Psi^D = \Psi_0 &= \phi_0 \wedge \diamond\Psi_1 \wedge \diamond\Psi_3 \\
&= \phi_0 \wedge \diamond(\phi_1 \wedge \psi_1 \wedge \diamond(\phi_2 \wedge \psi_2 \wedge \theta_2) \wedge \diamond(\phi_4 \wedge \psi_4) \wedge \diamond\theta_5) \wedge \\
&\quad \diamond(\phi_3 \wedge \diamond(\psi_6 \wedge \theta_6 \wedge \diamond(\phi_7 \wedge \psi_7 \wedge \theta_7) \wedge . \diamond \psi_8)).
\end{aligned}
$$

Lemma 3.5 (Testing Lemma). *For a FKD $D = (W_D, R_D)$ and a theory T, the following are equivalent:*

(1) *D is T-consistent, i.e., there exists a Kripke model of T witnessing D,*
(2) $T \not\models \neg\Psi^D$.

Proof. Recall that $T \not\models \neg\Psi^D$ iff there is a Kripke model $M = (W_M, \ldots)$ and a world $w \in W_M$ such that $M \models T$ but $(M, w) \models \Psi^D$.

- ($1 \Rightarrow 2$) Suppose $M = (W_M, R_M, \ldots)$ is a Kripke model which witnesses D via $f : W_D \to W_M$. Using induction on the tree structure of D, working from the leaves back to the root, we show that $(M, f(w_i)) \models \Psi_i$ for each $w_i \in W_D$.

- ∘ If w_i is a leaf in D: since M witnesses D, $(M, f(w_i)) \models \phi$ for each $\phi \in w_i$. But Ψ_i is merely the conjunction of all such ϕ, so $(M, f(w_i)) \models \Psi_i$.
- ∘ For all other w_i in D, we just have to show that $(M, f(w_i)) \models \diamond \Psi_j$ for each w_j such that $(w_i, w_j) \in R_D$. Since M witnesses D, $(f(w_i), f(w_j)) \in R_M$ for each such w_j; and by the inductive hypothesis, $(M, f(w_j)) \models \Psi_j$. Hence $(M, f(w_i)) \models \diamond \Psi_j$, as desired.

We have shown that $(M, f(w_i)) \models \Psi_i$ for each $w_i \in W_D$. In particular, $(M, f(w_0)) \models \Psi^D$. Since $M \models T$ by assumption, we have that $T \not\models \neg\Psi^D$.

- (2 ⇒ 1) Suppose that $T \not\models \neg\Psi^D$. Then there is a Kripke model $M = (W_M, R_M, \ldots)$ with some $w \in W_M$ such that $(M, w) \models \Psi^D$. We claim that M witnesses D via a function $f : W_D \to W_M$, which we define by induction. In this case we work from the root w_0 out to the leaves of tree, "unwinding" the nested diamonds in Ψ^D as we go. As we define f, we also show that $(M, f(w_i)) \models \Psi_i$ for each $w_i \in W_D$:
  - ∘ Base case ($i = 0$): For the root node w_0, set $f(w_0) = w$. Note that $(M, w) \models \Psi_0$ by assumption.
  - ∘ Inductive step: Take a world w_i which has a parent w_j, i.e., $(w_j, w_i) \in R_D$. By induction, $f(w_j)$ is a world in M such that $(M, f(w_j)) \models \Psi_j$. Note that Ψ_j is a conjunction which includes among its conjuncts $\diamond\Psi_i$ (since $(w_j, w_i) \in R_D$). Hence, $(M, f(w_j)) \models \diamond \Psi_i$, so there exists some $w' \in W_M$ such that $(f(w_j), w') \in R_M$ and $(M, w') \models \Psi_i$. Fix such a w' and set $f(w_i) = w'$. □

Since the representing formula Ψ^D allows us to test the T-consistency of a FKD D, we can use it to carry out the construction which proves the effective completeness theorem for modal logic.

3.5. The construction

Construction 3.6. Suppose T is a decidable theory in a first-order modal language L. Moreover, assume T has a Kripke model. The construction will produce a Kripke model M such that M is a decidable model of T.

Begin by fixing a computable set of new constants C and an effective enumeration $\phi_0, \phi_1, \phi_2, \ldots$ of all sentences in the extended language $L(T) \cup C$. Also fix a recursive function $\pi : \mathbf{N} \to \mathbf{N} \times \mathbf{N}$. In stage n we will compute $\pi(n) = (i, e)$, and then satisfy the completeness requirement with respect to ϕ_e at w_i in stage n, provided that w_i exists at stage n. To make sure that we satisfy each completeness requirement at each possible world, we fix π such that for each pair $(i, e) \in \mathbf{N} \times \mathbf{N}$ there are infinitely many $n \in \mathbf{N}$ with $\pi(n) = (i, e)$. Then, no matter when in the construction w_i is created, there will be a later stage n at which we satisfy the eth completeness requirement at w_i.

We will construct a sequence of FKDs $D_n = (W_n = \{w_0^n, \ldots, w_p^n\}, R_n)$, from which we will define a Kripke model M. An existing world w_i^n of D_n will correspond to a world w_i in the model M. The contents of w_i^n are the sentences that will be true at w_i in M; we will refer to them as the sentences associated with w_i (as of stage n).

Stage -1: $D_0 = (\{w_0^0 = \emptyset\}, R_0 = \emptyset)$.

Stage n: By induction, we have a FKD $D_n = (W_n = \{w_0^n, \ldots, w_p^n\}, R_n)$. Compute $\pi(n) = (i, e)$. If $i > p$, do nothing at this stage except update indices: put $D_{n+1} = (W_{n+1} = \{w_0^{n+1}, \ldots, w_p^{n+1}\}, R_{n+1})$, with $w_j^{n+1} = w_j^n$ (for $j = 0, \ldots, p$), $R_{n+1} = R_n$, and go on to stage $n + 1$.

If $i \leqslant p$, then we shall satisfy the eth completeness requirement at w_i. Let $D = D_n + \{\phi_e \in w_i^n\}$. Using the decidability of T, effectively check whether $T \models \neg\Psi^D$.

(1) If $T \models \neg\Psi^D$, let $D_{n+1} = (\{w_0^{n+1}, \ldots, w_p^{n+1}\}, R_{n+1})$ with $w_i^{n+1} = w_i^n \cup \{\neg\phi_e\}$; $w_j^{n+1} = w_j^n$ for $j \neq i$; and $R_{n+1} = R_n$, i.e., we associate $\neg\phi_e$ with w_i.

(Since this type of situation will arise repeatedly, we abbreviate it by saying: "set $D_{n+1} = D_n + \{\neg\phi_e \in w_i^n\}$ and update the indices from n to $n + 1$.")

(2) If $T \not\models \neg\Psi^D$, we associate ϕ_e with w_i. We also satisfy a Henkin witness or a $\diamond$-witness requirement for ϕ_e at w_i if necessary:

- if $\phi_e = \exists x\psi(x)$, let $D_{n+1} = D_n + \{\phi_e, \psi(c_j) \in w_i^n\}$ where c_j is the least element of C not occurring in D_n, and update indices. (So c_j is a Henkin witness for $\exists x\psi(x) \in w_i$.)
- if $\phi_e = \diamond\psi$, let $D_{n+1} = (\{w_0^{n+1}, \ldots, w_p^{n+1}, w_{p+1}^{n+1}\}, R_{n+1})$ where $w_i^{n+1} = w_i^n \cup \{\phi_e\}$; $w_{p+1}^{n+1} = \{\psi\}$; $w_j^{n+1} = w_j^n$ for $j \neq i, p+1$; and $R_{n+1} = R_n \cup \{(w_i, w_{p+1})\}$. (We have created a new world w_{p+1} which serves as a "$\diamond$-witness" for $\diamond\psi \in w_i$.)
- Otherwise, just let $D_{n+1} = D_n + \{\phi_e \in w_i^n\}$ and update indices.

It should be clear that the D_n form an increasing sequence, in two senses. Suppose $D_n = (W_n = \{w_0^n, \ldots, w_p^n\}, R_n)$, $D_m = (W_m = \{w_0^m, \ldots, w_q^m\}, R_m)$ with $n \leqslant m$. Then:

- the trees underlying the diagrams are increasing: $p \leqslant q$, and $(w_i^n, w_j^n) \in R_n$ implies $(w_i^m, w_j^m) \in R_m$
- the sets of sentences associated with w_i are increasing: $w_i^n \subseteq w_i^m$ for $i = 0, \ldots, p$.

Now we collect all the sentences associated with a world w_i throughout the construction. In keeping with the notation of the canonical model, we call this collection w_i. For each i such that w_i^n occurs in some D_n, let

$$w_i = \bigcup_{n=1}^{\infty} w_i^n.$$

Now we define the Kripke model $M = (W, R, D, I)$:

- the set of possible worlds is $W = \{w_i : w_i^n \text{ occurs in some } D_n\}$
- the possibility relation R is defined from the diagrams D_n in the obvious way:

$$(w_i, w_j) \in R \iff (w_i^n, w_j^n) \in R_n \quad \text{for some } n$$

- the domain D is the set of new constants C
- the interpretation I is defined from the syntactic contents of the w_i, just as in the canonical model: for an n-place relation symbol P, n-tuple $\bar{c}$ of elements of C, and possible world $w_i \in W$,

$$\bar{c} \in I(w_i)(P) \iff P(\bar{c}) \in w_i.$$

3.6. The consistency and closure lemmas

Lemma 3.7 (Consistency Lemma). *Each FKD D_n is T-consistent.*

Proof. By induction on n:

Base case: D_0 is T-consistent since T has a model by hypothesis.

Induction step: Assume D_n is T-consistent. To show that D_{n+1} is T-consistent, look at how D_n is extended to D_{n+1} at stage n. (Recall that we used the notation $D = D_n + \{\phi_e \in w_i^n\}$.)

(1) In the case that $T \models \neg\Psi^D$, we set $D_{n+1} = D_n + \{\neg\phi_e \in w_i^n\}$. By the Testing Lemma, no model of T witnesses D. Since D_n is T-consistent, there is some model M of T which witnesses D_n. Then $(M, f(w_i^n)) \not\models \phi_e$ (if it did, M would witness D). So $(M, f(w_i^n)) \models \neg\phi_e$, meaning that M also witnesses D_{n+1}.

(2) In the case that $T \not\models \neg\Psi^D$, we set $D_{n+1} = D_n + \{\phi_e \in w_i^n\}$. By the Testing Lemma, there is a model M of T which witnesses D. We claim that M also witnesses D_{n+1}. Note that since M witnesses D, $(M, f(w_i^n)) \models \phi_e$.

- if $\phi_e = \exists x\psi(x)$: since $(M, f(w_i^n)) \models \phi_e$, there is an element a in the domain of M such that $(M, f(w_i^n)) \models \psi(c_a)$. Then we can extend M to a model which witnesses D_{n+1} by putting $I(c_j) = a$.
- if $\phi_e = \diamond\psi$: since $(M, f(w_i^n)) \models \phi_e$, there is some world w of M such that $(f(w_i^n), w) \in R_M$ and $(M, w) \models \psi$. Then M witnesses D_{n+1} if we expand f by mapping the new w_{p+1}^{n+1} to some such "$\diamond$-witness" w.
- If ϕ_e is not an existential nor a diamond sentence, M witnesses $D_{n+1} = D_n + \{\phi_e \in w_i^n\}$ simply because $(M, f(w_i^n)) \models \phi_e$. $\square$

We wish to prove the Truth Lemma for M. To do so, we show that the w_i have certain closure properties.

Lemma 3.8 (Closure Lemma). *The following are true for each $w_i \in W$:*

(1) *For each sentence ϕ in the extended language $L \cup C$, exactly one of ϕ or $\neg\phi$ is in w_i;*
(2) $(\phi \wedge \psi) \in w_i \Leftrightarrow \phi \in w_i$ *and* $\psi \in w_i$;
(3) $\exists x\psi(x) \in w_i \Leftrightarrow$ *there exists* $c \in C$ *such that* $\psi(c) \in w_i$;
(4) $\diamond\psi \in w_i \Leftrightarrow$ *there exists* $w_j \in W$ *such that* $(w_i, w_j) \in R$ *and* $\psi \in w_j$;
(5) $\Box\psi \in w_i \Leftrightarrow \psi \in w_j$ *for every* $w_j \in W$ *such that* $(w_i, w_j) \in R$;
(6) $T \subseteq w_i$.

Proof.

(1) Find n large enough such that w_i^n exists in D_n, $n = (i, e)$, and $\phi = \phi_e$. Such an n exists by the properties π. By the construction, either $\phi_e \in w_i^{n+1}$ or $\neg\phi_e \in w_i^{n+1}$.

If both $\phi \in w_i$ and $\neg\phi \in w_i$, then they are both in some w_i^n. But this contradicts the T-consistency of D_n: clearly there is no model M with a world $f(w_i^n)$ such that $(M, f(w_i^n)) \models \phi$ and $(M, f(w_i^n)) \models \neg\phi$.

(2) ($\Rightarrow$) Suppose $(\phi \wedge \psi) \in w_i$ but $\phi \notin w_i$. By (1), $\neg\phi \in w_i$. Then $(\phi \wedge \psi), \neg\phi \in w_i^n$ for some n. But this contradicts the T-consistency of D_n. Hence, $\phi \in w_i$. Similarly for ψ.
   ($\Leftarrow$) Suppose $\phi, \psi \in w_i$ but $(\phi \wedge \psi) \notin w_i$. Then $\neg(\phi \wedge \psi) \in w_i$. This contradicts the T-consistency of some D_n.

(3) ($\Rightarrow$) Suppose $\exists x\psi(x) \in w_i$. Then $\exists x\psi(x)$ is added to w_i^n for some n. Then $\psi(c)$ is also added to w_i^n in the construction for some $c \in C$ (to fulfill the Henkin witness requirement).
   ($\Leftarrow$) Suppose $\psi(c) \in w_i$ for some $c \in C$ but $\exists x\psi(x) \notin w_i$. Then $\neg\exists x\psi(x) \in w_i$. This contradicts the T-consistency of some D_n.

(4) ($\Rightarrow$) Suppose $\diamond\psi \in w_i$. Then $\diamond\psi$ is added to w_i^n for some n. Then in the construction there is a newly existing world w_j^{n+1} in D_{n+1} with $\psi \in w_j^{n+1}$ and $(w_i^n, w_j^n) \in R_n$ (to fulfill the $\diamond$-witness requirement). Thus, there is a w_j as required.
   ($\Leftarrow$) Suppose there is a w_j such that $(w_i, w_j) \in R$ and $\psi \in w_j$, but $\diamond\psi \notin w_i$. Then $\neg \diamond \psi \in w_i$. So there is an n such that $\neg \diamond \psi \in w_i^n, (w_i^n, w_j^n) \in R_n$ and $\psi \in w_j^n$. This contradicts the T-consistency of D_n: clearly there is no model M with worlds $f(w_i^n)$ and $f(w_j^n)$ such that $(M, f(w_i^n)) \models \neg \diamond \psi, (f(w_i^n), f(w_j^n)) \in R$ and $(M, f(w_j^n)) \models \psi$.

(5) ($\Rightarrow$) Suppose $\Box\psi \in w_i$ but $\psi \notin w_j$ for some w_j such that $(w_i, w_j) \in R$. Then there is an n such that $\Box\psi \in w_i^n, (w_i^n, w_j^n) \in R_n$ and $\neg\psi \in w_j^n$. This contradicts the T-consistency of D_n.
   ($\Leftarrow$) Suppose $\Box\psi \notin w_i$. Then $\neg\Box\psi \in w_i$. Then $\diamond\neg\psi \in w_i$ also (if not, then the consistency of some D_n is violated). By (4), there is a w_j such that $(w_i, w_j) \in R$ and $\neg\psi \in w_j$, meaning $\psi \notin w_j$.

(6) Suppose $\phi \in T$ but $\phi \notin w_i$. Then $\neg\phi \in w_i$, and so $\neg\phi \in w_i^n$ for some n. But this contradicts the T-consistency of D_n: if M is a model of T which witnesses D_n, then $(M, f(w_i^n)) \models \phi$ (since $\phi \in T$) and $(M, f(w_i^n)) \models \neg\phi$ (since M supposedly witnesses D_n). $\Box$

3.7. The truth lemma

As with a canonical model, we have a Truth Lemma for the model M. Since we effectively enumerated the contents of the w_i in the construction, this will lead to the result we have been working towards—that M is a decidable model of T.

Lemma 3.9 (Truth Lemma). *For each $w_i \in W$ and each sentence ϕ of $L(T) \cup C$,*

$$(M, w_i) \models \phi \quad \Leftrightarrow \quad \phi \in w_i.$$

Proof. By induction on the structure of ϕ. The base case follows directly from the definition of M, while each of the other cases follows from the inductive hypothesis (IH) and the Closure Lemma (CL).

- ϕ atomic: by definition of M. The atomic relations at world w_i in M are defined precisely by the contents of w_i.
- $\phi = \psi \wedge \theta$: $(M, w_i) \models \psi \wedge \theta$ iff $(M, w_i) \models \psi$ and $(M, w_i) \models \theta$ iff $\psi \in w_i$ and $\theta \in w_i$ (IH) iff $\psi \wedge \theta \in w_i$ (CL).

- $\phi = \neg\psi$: $(M, w_i) \models \neg\psi$ iff $(M, w_i) \not\models \psi$ iff $\psi \notin w_i$ (IH) iff $\neg\psi \in w_i$ (CL).
- $\phi = \exists x\psi(x)$: $(M, w_i) \models \exists x\psi(x)$ iff $(M, w_i) \models \psi(c)$ for some $c \in C$ iff $\psi(c) \in w_i$ for some $c \in C$ (IH) iff $\exists x\psi(x) \in w_i$ (CL).
- $\phi = \diamond\psi$: $(M, w_i) \models \diamond\psi$ iff $(w_i, w_j) \in R$ and $(M, w_j) \models \psi$ for some $w_j \in W$ iff $(w_i, w_j) \in R$ and $\psi \in w_j$ for some $w_j \in W$ (IH) iff $\diamond\psi \in w_i$ (CL).
- $\phi = \Box\psi$: $(M, w_i) \models \Box\psi$ iff $(M, w_j) \models \psi$ for every $w_j \in W$ such that $(w_i, w_j) \in R$ iff $\psi \in w_j$ for every $w_j \in W$ such that $(w_i, w_j) \in R$ (IH) iff $\Box\psi \in w_i$ (CL). □

Corollary 3.10. $M \models T$.

Proof. By the Closure Lemma, $T \subseteq w_i$ for each $w_i \in W$. Then the Truth Lemma implies that $(M, w_i) \models \phi$ for every $\phi \in T$ and every $w_i \in W$. □

3.8. The effective completeness theorem

Theorem 3.11. *Every decidable theory in a modal language L has a decidable Kripke model.*

Proof. For a decidable theory T, let M be the Kripke model produced by Construction 3.6. We claim that M is a decidable model of T. We observed above that $M \models T$. Now we show that M is a decidable model.

- R is decidable: for any $w_i, w_j \in W$, to decide whether $(w_i, w_j) \in R$, search for the stage n of the construction in which w_j was created. Then $(w_i, w_j) \in R$ iff w_j was created as a $\diamond$-witness for w_i at that stage.
- truth is decidable: by the Truth Lemma, $(M, w_i) \models \phi$ iff $\phi \in w_i$, so we search for a stage at which the completeness requirement for ϕ is satisfied for w_i. That is, for any $w_i \in W$ and sentence ϕ in $L \cup C$, to decide whether $(M, w_i) \models \phi$ we find a stage n such that $\pi(n) = (i, e)$, w_i exists at stage n, and $\phi = \phi_e$. Then $(M, w_i) \models \phi$ iff ϕ_e is added to w_i at stage n. Since such an n can be found effectively, it is decidable whether $(M, w_i) \models \phi$. □

3.9. Global decidability

We have showed how to construct a decidable Kripke model of a given decidable modal theory. Our definition of a decidable Kripke model is a model M in which "truth at a world," or "local truth," is decidable: given a world w and a sentence ϕ, it is decidable whether $(M, w) \models \phi$. We now call this a *locally decidable* model.

We will alter the construction to produce a model M in which "truth in the model," or "global truth," is also decidable: given a sentence ϕ, it is decidable whether $M \models \phi$. In particular, given a decidable modal theory T, we shall construct a model M such that $M \models \phi$ iff $T \models \phi$.

Definition 3.12. A Kripke model $M = (W, R, D, I)$ is *globally decidable* if the sets W, D and the relation R are each computable, and truth in the model is decidable, i.e., the set

$$\{\phi : M \models \phi\}$$

is computable, where ϕ ranges over the sentences of L^D.

Recall that we constructed a locally decidable model by satisfying completeness requirements at each w_i. In stage n, we decided whether $(M, w_i) \models \phi_e$ or $(M, w_i) \models \neg\phi_e$, for $\pi(n) = (i, e)$. To construct a globally decidable model, we will include a stage for each sentence ϕ_e at which we will decide whether $M \models \phi_e$ or $M \not\models \neg\phi_e$. We will do this by simply checking whether $T \models \phi_e$. If so, we proceed as in the original construction; then $M \models \phi_e$, since we construct M so that it is a model of T. On the other hand, if $T \not\models \phi_e$, we ensure that $M \not\models \phi_e$ by creating a new world w_i at which $(M, w_i) \not\models \phi_e$. Unlike Construction 3.6, in which each new world was created as a successor a previously existing world, this new w_i will be "disconnected" from all previously existing worlds. This means that instead of tree FKDs, this construction will produce a sequence of "forest" FKDs, where each FKD consists of a (finite) set of trees.

Definition 3.13. (1) A FKD $D = (W, R)$ is a *forest FKD* if each connected component of the graph (W, R) is a tree FKD.

(2) A forest FKD is *locally T-consistent* if each connected component (considered as a FKD) has a witnessing model which is a model of T.

We will not need the forest FKDs to be T-consistent in the sense of Definition 3.2. Local T-consistency will suffice to prove the necessary closure properties and hence to define the model.

Construction 3.14. As in the Construction 3.6, we begin by fixing a new set of constants C, and an effective enumeration $\phi_0, \phi_1, \ldots$ of all sentences in the extended language $L \cup C$. We define a sequence of forest FKDs D_m as follows:

Stage -1: $D_0 = (\{w_0^0 = \emptyset\}, R_0 = \emptyset)$.

Stage $m = 2n + 1$: By induction we have a forest FKD D_m. Satisfy the eth completeness requirement at w_i, where $\pi(n) = (i, e)$; to do so, proceed as in stage n of Construction 3.6, using the representing formula of the tree FKD consisting of the connected component containing w_i. As Construction 3.6, this may include creating a new world as a $\diamond$-witness.

Stage $m = 2n + 2$: We have a forest FKD $D_m = (W = \{w_0^m, \ldots, w_p^m\}, R_m)$. Using the decidability of T, effectively check whether $T \models \phi_n$.

(1) If $T \models \phi_n$, set $D_{m+1} = D_m$ and update indices.
(2) If $T \not\models \phi_n$, we shall associate $\neg\phi_n$ with a new world w_{p+1}. Let $D_{m+1} = (\{w_0^{m+1}, \ldots, w_p^{m+1}, w_{p+1}^{m+1}\}, R_{m+1})$ where $w_i^{m+1} = w_i^m$ for $i = 0, \ldots, p$; $w_{p+1}^{m+1} = \{\neg\phi_n\}$; and $R_{m+1} = R_m$. Note that we do not extend the possibility relation in this case, so that the new world w_{p+1}^{m+1} is disconnected from all of the other existing worlds.

Now define a Kripke model M from this sequence of forest FKDs, in exactly the same way that we defined a model from the sequence of tree FKDs in Construction 3.6.

We have constructed the forest FKDs D_m based on criteria which maintain *local* T-consistency, so we have the following result.

Lemma 3.15 (Consistency Lemma). *Each D_m is locally T-consistent.*

Proof. By induction on m:

Base case: D_0 is T-consistent since T has a model by assumption.

Induction step: Assume D_m is locally T-consistent. To show that D_{m+1} is locally T-consistent, look at how D_m is extended to D_{m+1} at stage m:

Stage $m = 2n + 1$: In this stage we satisfy the eth completeness requirement at w_i, where $\pi(n) = (i, e)$, in the same way as in Construction 3.6. There we proved that this preserves T-consistency. Now we start with a locally consistent FKD $D_m = (W, R)$. So the component of (W, R) containing w_i^m has a witnessing model. By the same reasoning we provided in the original construction, the component has a witnessing model at the conclusion of this stage. Since we make no changes to any of the other components, D_{m+1} is also locally T-consistent.

Stage $m = 2n + 2$:

(1) In the case that $T \models \phi_n$, we do nothing except update indices. So clearly the local T-consistency of D_m implies D_{m+1} is locally T-consistent.
(2) In the case that $T \not\models \phi_n$, there is a model M of T with a world w such that $(M, w) \models \neg\phi_n$. Thus, the connected component of D_{m+1} containing w_{p+1}^{m+1} (which consists of just w_{p+1}) is witnessed by M via f, where $f(w_{p+1}) = w$. Since the other components of D_{m+1} are identical to those in D_m, D_{m+1} is locally T-consistent. □

The statements and proofs of the Closure Lemma and the Truth Lemma are exactly the same as before (Lemmas 3.8 and 3.9, respectively). We repeat the statements but omit the proofs:

Lemma 3.16 (Closure Lemma). *For each w_i,*

(1) *For each sentence ϕ, exactly one of ϕ or $\neg\phi$ is in w_i.*
(2) $(\phi \wedge \psi) \in w_i \Leftrightarrow \phi \in w_i$ *and* $\psi \in w_i$.
(3) $\exists x\psi(x) \in w_i \Leftrightarrow$ *there is a* $c \in C$ *such that* $\psi(c) \in w_i$.
(4) $\diamond\psi \in w_i \Leftrightarrow$ *there is a* w_j *such that* $(w_i, w_j) \in R$ *and* $\psi \in w_j$.
(5) $\Box\psi \in w_i \Leftrightarrow \psi \in w_j$ *for every* $w_j \in W$ *such that* $(w_i, w_j) \in R$.
(6) $T \subseteq w_i$.

Lemma 3.17 (Truth Lemma). *For each $w_i \in W$ and sentence ϕ of $L(T) \cup C$,*

$$(M, w_i) \models \phi \quad \Leftrightarrow \quad \phi \in w_i.$$

Finally, the following establishes that M is a globally decidable model:

Theorem 3.18. *$M \models \phi$ iff $\phi \in T$.*

Proof. ($\Leftarrow$) Suppose $\phi \in T$. By the Closure Lemma, $T \subseteq w_i$ for each $w_i \in W$. Hence, $\phi \in w_i$ for each $w_i \in W$. By the Truth Lemma, $(M, w_i) \models \phi$ for each $w_i \in W$, i.e., $M \models \phi$.

($\Rightarrow$) Suppose $\phi \notin T$, i.e., $T \not\models \phi$. Find n such that $\phi = \phi_n$. Then, according to stage $m = 2n + 2$ of the construction, D_{m+1} contains an existing world w^{m+1}_{p+1} such that $\neg\phi_n \in w^{m+1}_{p+1}$. By the Truth Lemma, $(M, w_{p+1}) \models \neg\phi$, so $M \not\models \phi$. □

Theorem 3.19. *Every decidable modal theory has a globally decidable model.*

Note that the model M produced by Construction 3.14 is also *locally* decidable, by the same arguments given in the proof of Theorem 3.11. Thus, we have proved that every decidable modal theory has a Kripke model which is both locally decidable and globally decidable.

4. The testing lemma: a deduction theorem on finite Kripke diagrams for modal logic

We have described the construction of a decidable Kripke model for a decidable modal theory. We generalized the effective predicate Henkin construction. To do so, we replaced the classical Deduction Theorem, which works with a statement as a premise, by the Testing Lemma, which essentially has a finite approximation to a Kripke model (a FKD) as a premise.

The predicate construction proceeded by effectively satisfying completeness requirements: a complete diagram of a model is built up by adding either ϕ or $\neg\phi$ for each sentence ϕ. The Deduction Theorem was used to decide *how* to satisfy each such requirement. Given a finite approximation to a complete diagram, we used the Deduction Theorem to decide whether to extend it by ϕ or $\neg\phi$.

The modal construction also proceeded by effectively satisfying completeness require ments—at each possible world that is constructed. The tools used to decide how to do so were the representing formula of a FKD and the Testing Lemma. Now, given a finite approximation to a model (a FKD), we use the Testing Lemma to determine whether to extend it by ϕ or $\neg\phi$ at a given possible world.

Thus, the predicate Deduction Theorem is replaced by the modal Testing Lemma in the (effective) construction of a model. This functional similarity is evidence for treating the Testing Lemma as the analogue, for modal logic, of the classical Deduction Theorem. In this section we present further evidence for this view.

First, we will rephrase the Deduction Theorem for predicate logic. Then we will introduce a tableau proof theory for modal logic, and adapt it for proofs within the context of FKDs. This lets us extend the Testing Lemma to include a syntactic aspect that deals with deductions. Finally, we examine how this extended version of the Testing Lemma can be seen as a generalization of the Deduction Theorem to modal logic.

4.1. The deduction theorem for classical logic

We begin by stating the Deduction Theorem (DT) for classical first-order logic. Earlier, when we used the DT in the classical first-order Henkin construction, we cited just its semantic (model-theoretic) aspect. Now we add syntactic (proof-theoretic) content. Then we rephrase the classical Deduction Theorem in an equivalent form which will show its relationship with the modal Testing Lemma.

Theorem 4.1 (Deduction Theorem for classical first-order logic). *Fix a classical first-order language L. For a theory T, a finite set of sentences Δ, and a sentence ϕ (all in the language L), the following are equivalent*:

(1) $T \cup \Delta \models \phi$;
(2) $T \models \Delta \to \phi$;
(3) $T \vdash \Delta \to \phi$;
(4) $T \cup \Delta \vdash \phi$.

(Note that in (2) and (3) we let Δ represent the conjunction of that finite set of sentences.)

Now let us make some modifications to the statement of the DT, with the intent of bringing the form of the DT closer to that of the Testing Lemma (as stated in Lemma 3.5). First, we replace ϕ by $\neg\phi$. So now the clauses of the DT are:

(1) $T \cup \Delta \models \neg\phi$;
(2) $T \models \Delta \to \neg\phi$;
(3) $T \vdash \Delta \to \neg\phi$;
(4) $T \cup \Delta \vdash \neg\phi$.

The next modification is more significant. In Section 3.3, we motivated the idea of the representing formula of a FKD by replacing the implication $\Delta \to \neg\phi$ in the DT by the equivalent $\neg(\Delta \wedge \phi)$.

Finally, we negate each of the clauses in the DT, i.e., we replace $\models$ by $\not\models$ in clauses (1) and (2), and $\vdash$ by $\nvdash$ in clauses (3) and (4). The DT now states that TFAE:

(1) $T \cup \Delta \not\models \neg\phi$;
(2) $T \not\models \neg(\Delta \wedge \phi)$;
(3) $T \nvdash \neg(\Delta \wedge \phi)$;
(4) $T \cup \Delta \nvdash \neg\phi$.

Now we can begin to see the connection between the classical Deduction Theorem and the modal Testing Lemma. First, think of the finite set of sentences Δ as the finite diagram of a classical model: a "classical finite diagram." Recall that this is exactly how the DT is used in the classical Henkin construction, as we noted in our discussion in Section 3.3. Then the conjunction $\Delta \wedge \phi$ in clauses (2) and (3) of the DT is the formula representing Δ extended by ϕ. We are justified in calling the conjunction $\Delta \wedge \phi$ the representing formula if we think of such a classical finite diagram as a FKD with a single existing world. Then, according to Definition 3.4, the representing formula of such a FKD is indeed the conjunction of the formulas at that single world.

On the other hand, Ψ^D in clause (2) of the Testing Lemma is the representing formula of a FKD D. So if we think of the finite classical diagram $\Delta \cup \{\phi\}$ as corresponding to a finite Kripke diagram D, both clauses (2) say the same thing: that the negation of the representing formula of a finite diagram is not a consequence of T. Or, in other words, that the representing formula of the finite diagram is *satisfiable* with respect to T.

Meanwhile, clause (1) of the DT says that $\neg\phi$ is not a consequence of $T \cup \Delta$, i.e., that there is a model of $T \cup \Delta$ which also satisfies ϕ. To see how this relates to the modal Testing Lemma, note that this is the same as saying that there is a model of T which satisfies, or "witnesses," $\Delta \cup \{\phi\}$. Once again, we are justified in using this terminology we introduced if we think of $\Delta \cup \{\phi\}$ as a FKD with a single world. Then a witnessing model is a Kripke model with some world which satisfies all of $\Delta \cup \{\phi\}$; this world in the witnessing model yields a classical model which satisfies $\Delta \cup \{\phi\}$. Thus, clause (1) of the DT corresponds to clause (1) of the Testing Lemma: both say that there is a model of T which witnesses a finite diagram.

In summary, the Testing Lemma is a modal version of the equivalence of (1) and (2) above, which is the semantic part of the classical DT. This is the first evidence that the Testing Lemma should be treated as a Deduction Theorem for modal logic. Next we will add clauses (3) and (4) to the modal Testing Lemma. For these we need a proof theory.

4.2. Modal tableau proofs

Historically, modal logics were first examined from a syntactic viewpoint, i.e., proof-theoretically, prior to the development of possible worlds semantics. For example, the well-known modal logics **K**, **T**, **S4**, etc. were originally formulated as axiomatic systems.

Axiomatic proof systems for modal logics continue to be used and studied. Subsequent to the development of possible worlds semantics, however, an alternative style of proof theory was developed: tableau systems. Tableau proofs for modal logics differ significantly from axiomatic proofs, in that the tableaux make explicit mention of possible worlds and a possibility relation. There are two kinds of entries on a modal tableau:

- *truth statements* of the form $Tw \models \phi$ or $Fw \models \phi$, which is meant to assert that a formula ϕ is true or false in a possible world w;
- *accessibility statements* of the form wRw', which is meant to assert that a world w' is accessible from another world w.

On the other hand, axiomatic proof systems for modal logics make no mention of possible worlds or a possibility relation. Entries in an axiomatic proof are just formulas, detached from any possible worlds or a possibility relation. In terms of the semantics, this is because the entries in an axiomatic proof are supposed to hold *globally*, at every possible world.

By contrast, we wish to work with proofs within FKDs, which consist of formulas associated with specific existing worlds, and a possibility relation between those worlds.

Thus, tableau proofs are better suited for our purposes, since they possess precisely the structure needed to work with FKDs syntactically.

For the basics of modal tableau proofs, we refer to Chapter 4 of [10]. In particular, we will use the definitions of the following:

- modal tableau;
- contradictory path, tableau;
- tableau proof;
- tableau from (global) premises;
- complete systematic tableau.

Consult also [3,4] for further information on tableau proofs for modal logic.

4.3. Tableau deductions from a FKD

We now adapt modal tableaux for constructing proofs using FKDs, with the goal of using tableau deductions in our construction of a Kripke model. At each stage of the construction, we have to determine whether we can add a sentence ϕ to an existing world w_i of a FKD D. In Section 3 we described a way to do this semantically, using the concepts of witnessing models and representing formulas. Now we will describe a way to do this syntactically, using the concept of tableau deductions from a FKD.

For example, consider the FKD D of (Fig. 4).

Suppose we want to satisfy the completeness requirement with respect to ψ at w_1, i.e., we want to add either ψ or $\neg\psi$ to w_1.

We can see semantically that we cannot put $\psi \in w_1$: there cannot be a witnessing model M in which $(M, f(w_1)) \models \phi \wedge \psi$, $(M, f(w_0)) \models \neg\diamond(\phi\wedge\psi)$, and $(f(w_0), f(w_1)) \in R$. This is reflected by the fact that the representing formula $\Psi^{D+\{\psi\in w_1\}} = \neg\diamond(\phi\wedge\psi)\wedge\diamond(\phi\wedge\psi)$ is not satisfiable, i.e., $\models\neg\Psi^{D+\{\psi\in w_1\}}$.

Now we can make the argument syntactically, with a tableau proof:

Here, entries (1)–(3) are from the original FKD D, while entry (4) is the added assumption $\psi \in w_1$, which we are testing. From these premises we produce a contradictory tableau. We shall see that this *proves* that $D + \{\psi \in w_1\}$ has no witnessing model.

Or, if we add $Fw_1 \models \neg\psi$ at the root as entry (0), then we have a tableau proof of $w_1 \models \neg\psi$ from premises (1)–(3) of the FKD D. Entry (4) arises from developing the root entry (0) with the $(F\neg)$ rule.

This indicates how to use tableau deductions within the construction. We must use a new kind of tableau deduction, in which premises are provided by a FKD.

First, we define the *stem* of a tableau τ as the set of entries which appear on every path through τ. Intuitively, we think of the stem as consisting of the root plus the

$$w_0 = \{\neg\Diamond(\phi\wedge\psi)\} \longrightarrow w_1 = \{\phi\}$$

Fig. 4. A simple FKD.

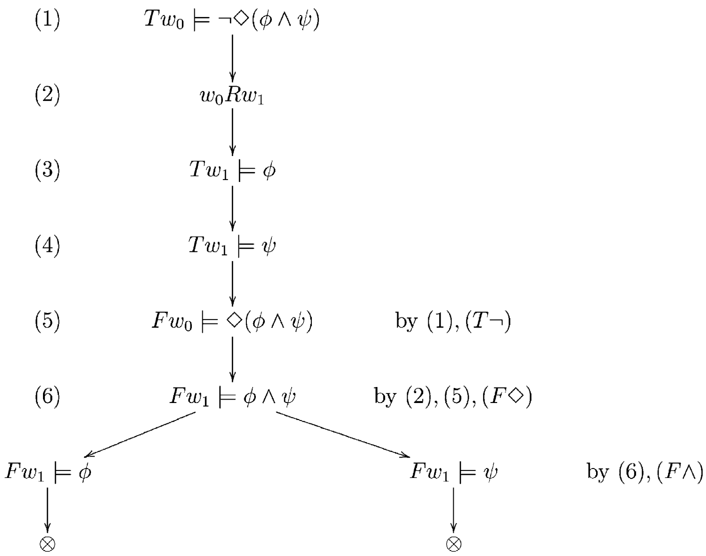

Fig. 5. A tableau deduction from a FKD.

entries which lie in a straight line below the root, above the first branching in the tableau. In the tableau in Fig. 5, the stem consists of entries (1)–(6).

Definition 4.2. Let $D = (W_D = \{w_0, \dots, w_n\}, R_D)$ be a FKD.

(1) The *entries from* D consists of the following set of tableau entries:

$$\{Tw_i \models \phi \mid w_i \in W_D, \phi \in w_i\} \cup \{w_i R w_j \mid (w_i, w_j) \in R_D\}.$$

(2) A *tableau from* D is any tableau which contains on its stem all the entries from D.

As we noted earlier, tableau entries have exactly the right structure for expressing the contents of a FKD. If $\phi \in w_i$, we think that ϕ is true at a world w_i. That is expressed by a tableau entry $Tw_i \models \phi$. And clearly the entries $w_i R w_j$ represent the binary relation R_D within a tableau. For reasoning within a FKD D, we want all this information—namely, all of the entries from D—to be present on the stem.

In Fig. 5, entries (1)–(3) constitute the entries from D. Since these are on the stem, we have a tableau from D. Note that, as in this example, a tableau from D may have other entries besides the entries from D on its stem.[4]

[4] An alternative definition would follow the definition of a tableau from global premises. Instead of requiring that all the entries from D be on the stem, we would allow any path to be extended by any entry from D. The given definition of course encompasses this alternative one, since we can always repeat any entry from the stem along any path. But with the alternative definition we would have to specify that the "new" worlds and constants introduced by developing $(T\Diamond), (F\Box), (T\exists), (F\forall)$ entries are new with respect to D. By placing all of D on the stem, we guarantee that the new worlds and constants will be new to D.

Definition 4.3. Suppose $D=(W_D,R_D)$ be a FKD. For $w_i\in W$ and a formula ϕ, *a tableau deduction of* $w_i\models\phi$ *from* D is a contradictory tableau from D with root $Fw_i\models\phi$.

We can allow the presence of a set of "global" premises T in the standard way, which will be needed for our construction of a Kripke model of a theory T. Thus, we will be interested in what we call tableau deductions from (a FKD) D and (a theory) T.

Now we can sketch the connection between such tableau deductions and the semantic consistency of FKDs. Suppose we have produced a FKD D_n as in Construction 3.6 such that there is a tableau deduction of $w_i\models\neg\phi$ from D and T. We will show that such a tableau deduction establishes syntactically that $D+\{\phi\in w_i\}$ is not T-consistent. On the other hand, if $w_i\models\neg\phi$ is *not* tableau deducible from D and T, then the complete systematic tableau from D and T with root $Fw_i\models\neg\phi$ is noncontradictory. A noncontradictory path through this tableau yields a countermodel of T which witnesses $D+\{\phi\in w_i\}$. Thus, tableau deductions are another way of completing the "key step" of the construction, as discussed in Section 3.3.

4.4. The testing lemma revisited

We have sketched a proof that the following are equivalent:

- $D+\{\phi\in w_i\}$ is T-consistent;
- $w_i\models\neg\phi$ is not tableau deducible from D and T.

The first clause here is also the first clause in the Testing Lemma, and it is precisely the condition we need to test in the construction of a Kripke model. So now we have a syntactic clause to add to the Testing Lemma. We will see that it matches up with the last clause of the classical Deduction Theorem: $T\cup\Delta\nvdash\neg\phi$. So the remaining line of the classical Deduction Theorem to account for within the modal context is $T\nvdash\neg(\Delta\wedge\phi)$.

Recall that we looked at $(\Delta\wedge\phi)$ as the formula representing the extended finite diagram $\Delta+\{\phi\}$. In the modal case, we have the extended FKD $D+\{\phi\in w_i\}$, which has representing formula $\Psi^{D+\{\phi\in w_i\}}$. This leads us to the final clause of the modal Testing Lemma: $T\nvdash\neg\Psi^{D+\{\phi\in w_i\}}$. So we now have an extended version of the Testing Lemma:

Theorem 4.4 (Testing Lemma). *Fix a first-order modal language L. For a theory T, a tree FKD D, and a sentence* ϕ (*all in the language L*), *the following are equivalent*:

(1) $D+\{\phi\in w_i\}$ *is T-consistent*, *i.e.*, *there exists a Kripke model of T witnessing* $D+\{\phi\in w_i\}$;
(2) $T\not\models\neg\Psi^{D+\{\phi\in w_i\}}$;
(3) $T\nvdash\neg\Psi^{D+\{\phi\in w_i\}}$;
(4) $w_i\models\neg\phi$ *is not tableau deducible from D and T*.

Proof. $(1) \Leftrightarrow (2)$: This was part of the original Testing Lemma (Lemma 3.5).

$(2) \Rightarrow (3)$: By the soundness of tableau provability (see Section IV.4 of [10]).

$(4) \Rightarrow (1)$: This follows the proof of completeness of tableau provability, as we sketched at the end of the previous section. Assume $w_i \models \neg\phi$ is not tableau deducible from D and T. Then the complete systematic tableau (CST) from D and T with root $Fw_i \models \neg\phi$ contains a noncontradictory path. This noncontradictory path yields a Kripke model. By definition of the CST, it is a model of T. Since $Fw_i \models \neg\phi$ and the entries from D are on the path, the model witnesses $D + \{\phi \in w_i\}$.

$(3) \Rightarrow (4)$: We prove the contrapositive. Suppose that $w_i \models \neg\phi$ is tableau deducible from D and T. Then there is a contradictory tableau τ with root $Fw_i \models \neg\phi$ and the entries from D on the stem. We will show that $T \vdash \neg\Psi^{D+\{\phi \in w_i\}}$, by constructing a tableau τ' which "mimics" the tableau proof τ.

The root of τ' is of course $Fw_0 \models \neg\Psi^{D+\{\phi \in w_i\}}$. As it is child we put $Tw_0 \models \Psi^{D+\{\phi \in w_i\}}$. Next we "unwind" the representing formula $\Psi^{D+\{\phi \in w_i\}}$, using the tableau development rules. This will produce the entries from $D + \{\phi \in w_i\}$ on the stem of τ'. From that point on we can mimic τ.

More precisely, suppose $D' = D + \{\phi \in w_i\} = (W, R)$. We use induction on the tree structure of (W, R) to construct τ' so that for each $w_k \in W$

- $\psi \in w_k \Rightarrow Tw_k \models \psi$ is on the stem of τ;
- $(w_j, w_k) \in R \Rightarrow w_j R w_k$ is on the stem of τ;
- $(w_k, w_l) \in R \Rightarrow Tw_k \models \diamond\Psi_l$ is on the stem of τ.

The first two clauses will establish that the entries from D' are on the stem of τ'. The entries $Tw_k \models \diamond\Psi_l$ of the last clause also appear on the stem as we unwind $\Psi^{D'}$, and will be useful as part of the induction hypothesis.

Our induction works from the root node w_0 of D' out to the leaves. For the base case, recall that $\Psi_0 = \Psi^{D'} = \bigwedge(\{\psi \mid \psi \in w_0\} \cup \{\diamond\Psi_l \mid (w_0, w_l) \in R\})$. So beginning with $Tw_0 \models \Psi^{D'}$, repeated applications of $(T\wedge)$ yield the entries $Tw_0 \models \psi$ for each $\psi \in w_0$ and $Tw_0 \models \diamond\Psi_l$ for each $w_l \in W$ such that $(w_0, w_l) \in R$. Since w_0 is the root, there is no w_j such that $(w_j, w_0) \in R$. Thus, we have established the base case.

For the induction step, take $w_k \in W$ with parent w_j, i.e., $(w_j, w_k) \in R$. By the induction hypothesis, $Tw_j \models \diamond\Psi_k$ is on the stem. So by applying the $(T\diamond)$ tableau development rule, we add entries $w_j R w_k$ and $Tw_k \models \Psi_k$ to the stem (here we choose w_k as the name of the new world in this application of the $(T\diamond)$ rule). Now, as in the base case, repeated applications of $(T\wedge)$ on $Tw_k \models \Psi_k$ yield the entries $Tw_k \models \psi$ for each $\psi \in w_k$ and $Tw_k \models \diamond\Psi_l$ for each w_l such that $(w_k, w_l) \in R$. This is depicted in Fig. 6.

Thus, from unwinding $Tw_0 \models \Psi^{D'}$ as described, we get a set of entries on the stem of τ'. Let us call this set of entries ρ. We have shown that ρ contains the entries from $D' = D + \{\phi \in w_i\}$ (plus other entries that arise from unwinding $\Psi^{D'}$).

Now that we have all the entries from $D' = D + \{\phi \in w_i\}$ on the stem, we define the rest of τ' by copying the development of τ below the entries from D on its stem. In particular, we may assume τ has structure shown in Fig. 7, for some tree of entries σ.

We now show that how to continue to produce τ' so that it has the structure shown in Fig. 8, where σ is the subtree of τ.

$$
\begin{array}{cl}
\vdots & \\
\downarrow & \\
Tw_0 \models \Diamond\Psi_k & \\
\downarrow & \\
w_j R w_k & \text{by } (T\Diamond) \\
\downarrow & \\
Tw_k \models \Psi_k & \\
\downarrow & \\
\vdots & \text{by } (T\wedge) \\
\downarrow & \\
Tw_k \models \psi & \text{for each } \psi \in w_k \\
\downarrow & \\
\vdots & \text{by } (T\wedge) \\
\downarrow & \\
Tw_k \models \Diamond\Psi_l & \text{for each } w_l \text{ such that } (w_k, w_l) \in R_D
\end{array}
$$

Fig. 6. Unwinding $\Diamond\Psi_k$.

$$
\begin{array}{c}
Fw_i \models \neg\phi \\
\downarrow \\
D \\
\downarrow \\
\sigma
\end{array}
$$

Fig. 7. Tableau proof τ.

$$
\begin{array}{c}
\rho \\
\downarrow \\
\sigma
\end{array}
$$

Fig. 8. Tableau deduction τ'.

We can prove this by a simple induction. Any entry E on σ appears from developing either

(1) the root $Fw_i \models \neg\phi$;
(2) an entry E' from D, or;
(3) a previous entry E' on σ.

$$\begin{tikzcd} Fw_i \models \neg\phi \arrow[d] \\ D \arrow[d] \\ P' \cap \sigma \end{tikzcd}$$

Fig. 9. The path P.

Case (1): the entry E must be $Tw_i \models \phi$. But this entry is already on ρ, since it is an entry from $D' = D + \{\phi \in w_i\}$, so we can repeat it.

Case (2): we have already established that the entries from D are on ρ, so we can develop E' as needed to produce E on τ'.

Case (3): by induction E' is on τ', so we can develop it to produce E on τ'.

Finally, we prove that τ' is contradictory. Take a path P' through τ'. Consider the corresponding path P through τ, as shown in Fig. 9.

Since τ is contradictory, there are entries $Tw \models \psi$ and $Fw \models \psi$ on P. We know the entries from D lie on the stem of τ'. So if the contradictory entries lie on $D \to P' \cap \sigma$ they also lie on P'.

The only remaining case is if the contradiction on P is caused by the root, i.e., P is contradictory because of entries $Fw_i \models \neg\phi$ and $Tw_i \models \neg\phi$. Then $Tw_i \models \neg\phi$ is on $D \to P' \cap \sigma$, so it also on P'. By developing it, we can assume $Fw_i \models \phi$ is on P'. Now recall that $Tw_i \models \phi$ is also on P', since it is an entry from D'. Hence, P' is contradictory, as desired. $\square$

4.5. The testing lemma and the deduction theorem

Finally, we compare the modal Testing Lemma we have just proved with the classical Deduction Theorem. We will see how the Testing Lemma can be viewed as a translation of the Deduction Theorem into modal logic.

For convenience, we repeat the two results:

Theorem 4.5 (Classical deduction theorem). *For a theory T, a finite set of sentences Δ, and a sentence ϕ (all in a classical first-order language L), the following are equivalent*:

(1) $T \cup \Delta \not\models \neg\phi$;
(2) $T \not\models \neg(\Delta \wedge \phi)$;
(3) $T \nvdash \neg(\Delta \wedge \phi)$;
(4) $T \cup \Delta \nvdash \neg\phi$.

Lemma 4.6 (Modal testing lemma). *For a theory T, a tree FKD D, and a sentence ϕ (all in modal language L), the following are equivalent*:

(1) $D + \{\phi \in w_i\}$ *is T-consistent, i.e., there exists a Kripke model of T witnessing* $D + \{\phi \in w_i\}$;

(2) $T \not\models \neg \Psi^{D+\{\phi \in w_i\}}$;
(3) $T \nvdash \neg \Psi^{D+\{\phi \in w_i\}}$;
(4) *$w_i \models \neg\phi$ is not tableau deducible from D and T.*

We usually think that the Deduction Theorem fails for modal logic, because we think that the finite set of classical sentences Δ must translate to a finite set of modal sentences. Here, we choose to view Δ as the finite diagram of a classical model. Its counterpart in modal logic is a FKD D, which adds the extra structure of possible worlds and a possibility relation.

With that view, both clauses (1) state the condition that needs to be checked in a Henkin-style construction of a model of a theory T: that there is a model of T which witnesses a particular extended diagram. As we explained earlier, clause (1) of the classical Deduction Theorem says that there is a model of T witnessing $\Delta \cup \{\phi\}$. Thus, in the classical case, the finite diagram Δ is extended by adding ϕ, to get $\Delta \cup \{\phi\}$. In the modal case, the finite diagram D is extended by adding ϕ at a particular existing world w_i, to get $D + \{\phi \in w_i\}$.

Thus, the modal Testing Lemma has the same use as the classical Deduction Theorem: both are used in the construction of a model to determine whether there is a witnessing model of an extended finite diagram, and thus if the diagram can be extended as such.

Initially it may have appeared that the Testing Lemma and the Deduction Theorem carry out this similar function in rather different ways. But by rephrasing the Deduction Theorem as we have, we can see further parallels.

Instead of viewing the Deduction Theorem as being about the implication $\Delta \to \neg\phi$, we view it as a statement about *conjunction*; namely, the conjunction $\Delta \wedge \phi$, which represents the extended finite diagram $\Delta \cup \{\phi\}$ syntactically. In the modal context, the representing formula of a FKD cannot be simply a conjuction, since it must represent the possibility relation of the FKD. The representing formula $\Psi^{D+\{\phi \in w_i\}}$ does this by nesting conjunctions within $\diamond$ modalities.

Then clauses (2) of both theorems state that the negation of the representing formula of the extended diagram is not a logical consequence of the theory T, i.e., that the representing formula is satisfiable. Similarly both clauses (3) state that this same negation is not provable from the theory T.

Finally, clause (4) is in both cases a proof-theoretic version of clause (1). In the classical case, it states that $\neg\phi$ is not deducible from the theory T and the finite diagram Δ. Similarly, in the modal case it states that $w_i \models \neg\phi$ is not (tableau) deducible from T and the FKD D.

We have argued that the Testing Lemma that we have proved for tree FKDs should be viewed as an analogue of the classical Deduction Theorem for modal logic. We began by noting the functional similarity between the two: the Testing Lemma is used in our effective construction of a Kripke model in the same way that the classical Deduction Theorem is used in the effective construction of a classical model. Then, after rephrasing the classical Deduction Theorem and introducing the notion of tableau deductions from a FKD, we demonstrated a strong formal similarity between the two theorems as well.

We have proved the Testing Lemma for a particular modal logic: first-order constant domain **K**. But because it uses very basic aspects of Kripke models and the $\diamond$ modality, the Testing Lemma carries over to a number of other modal logics. In the following Sections, we will use tree FKDs and Testing Lemmas to carry out effective constructions of Kripke models for these modal logics as well.

5. Special possibility relations

In Section 3, we proved an Effective Completeness Theorem for a particular modal logic. To effectively construct a Kripke model, we proved a Testing Lemma, and subsequently argued in Section 4 that this result can be viewed as the analogue of the classical Deduction Theorem for (this) modal logic.

In this section and the following one, we examine some common modal logics which are very similar to the one we have been working with. They contain the same $\Box$ and $\diamond$ modalities as before; but they formalize different interpretations of these modalities, through the following mechanisms:

- special types of possibility relations in the Kripke models;
- certain axiom schema, in axiomatic proof systems;
- certain tableau rules, in tableau proof systems.

We will show that for these logics, we can adapt the tools and techniques we have developed in Sections 2 and 3 just slight modifications.

5.1. *The logics*

The modal logic we have been working with up to this point, as defined in Section 2.2, is usually referred to as **K**. Formally, we can view **K** as either

- the set of valid sentences (the sentences true in every world of every Kripke model);
- the set of tableau provable sentences (the sentences ϕ such that there is a contradictory tableau with root $Fw \models \phi$).

By the Completeness Theorem for tableau proofs, these two sets coincide.

As we mentioned in Section 4, axiomatic treatments of modal logics preceeded both possible worlds semantics and tableau proof systems. So the original definition of **K** was actually as the following axiomatic system:

- axiom schema from an axiomatic system for classical predicate logic;
- axiom scheme: $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$;
- rule of inference (modus ponens) : from ϕ and $\phi \to \psi$ infer ψ;
- rule of inference (generalization) : from ϕ infer $\Box\phi$.

The set of theorems of this axiomatic system coincides with the definition of **K** in terms of tableau provability and valid sentences (see [8] for a proof of this fact).

K is often viewed as a starting point, which can be extended by adding various axiom schema. Some common axiom schema, with their traditional names, are:

- $T : \Box\phi \to \phi$;
- $4 : \Box\phi \to \Box\Box\phi$;
- $5 : \neg\Box\phi \to \Box\neg\Box\phi$.

The following axiomatic systems define some common modal logics:

- **T** consists of **K** and the axiom scheme T;
- **S4** consists of **T** and the axiom scheme 4 (i.e., **K** and the axiom schema T and 4);
- **K4** consists of **K** and the axiom scheme 4;
- **K5** consists of **K4** and the axiom scheme 5 (i.e., **K** and the axiom schema 4 and 5);
- **S5** consists of **S4** and the axiom schema 5 (i.e., **K** and the axiom schema T, 4, and 5).

For the rest of this section, we will let $\mathscr{L}$ stand for any of these six logics: **K**, **T**, **K4**, **S4**, **K5** or **S5**.

Subsequent to the definition of these logics as axiomatic systems, it was discovered that they have very natural characterizations, both in terms of the possible worlds semantics and in terms of tableau provability. The set of theorems of each axiomatic system above coincides with the set of formulas valid with respect to a natural class of Kripke models, and also with the set of sentences provable using a certain set of tableau rules. In each case, we call the corresponding class of models the $\mathscr{L}$-models, and the corresponding set of tableau rules the $\mathscr{L}$-rules.

Definition 5.1. (1) A Kripke model $M=(W,R,D,I)$ is a (**T**, **K4**, **S4**, **K5**, **S5**)-model iff the possibility relation R is (reflexive, transitive, reflexive and transitive, transitive and Euclidean, an equivalence relation).

(2) The (**T**, **K4**, **S4**, **K5**, **S5**)-rules consist of the basic tableau rules plus the (reflexive, transitive, reflexive and transitive, transitive and Euclidean, complete) tableau development rule(s). (See Section IV.5 of [10] for the definitions of these tableau development rules.)

In addition, we say every Kripke model is a **K**-model, and we let the **K**-rules refer to just the basic tableau rules.

With this notation in place, we say a sentence ϕ is

- *valid with respect to the $\mathscr{L}$-models* (notation: $\models_{\mathscr{L}} \phi$) if $M \models \phi$ for every $\mathscr{L}$-model M;
- *tableau provable using the $\mathscr{L}$-rules* (notation: $\vdash_{\mathscr{L}} \phi$) if there is a contradictory tableau generated using the $\mathscr{L}$-rules with root $Fw \models \phi$.

This notation is justified by the following result.

Theorem 5.2. *For each logic* $\mathscr{L}$,

$$\mathscr{L} = \{\phi \mid \models_{\mathscr{L}} \phi\} = \{\phi \mid \vdash_{\mathscr{L}} \phi\}.$$

Proof. See Section IV.5 of [10]. □

For example, the logic **T** coincides with the set of sentences valid with respect to reflexive Kripke frames. It also coincides with the set of sentences for which there are tableau proofs using the basic tableau rules plus the reflexive tableau development rule.

To state an effective completeness theorem for each logic $\mathscr{L}$, we need to extend these notions of validity and provability to include a set of (global) premises T. We can then formulate the concept of a theory with respect to $\mathscr{L}$.

Definition 5.3. For any set of sentences T and each logic $\mathscr{L}$,

- $T \models_{\mathscr{L}} \phi \Leftrightarrow$ if M is a $\mathscr{L}$-model of T, then $M \models \phi$;
- $T \vdash_{\mathscr{L}} \phi \Leftrightarrow \phi$ is tableau provable from T using $\mathscr{L}$-rules.

The Completeness Theorem(s) connect these notions:

Theorem 5.4. $T \models_{\mathscr{L}} \phi \Leftrightarrow T \vdash_{\mathscr{L}} \phi$.

Proof. Again, see [10]. □

Definition 5.5. A set of sentences T is an $\mathscr{L}$*-theory* iff it is closed with respect to $\models_{\mathscr{L}}$ (or equivalently, $\vdash_{\mathscr{L}}$).

5.2. *Testing lemmas*

We wish to generalize the techniques developed for **K** in Section 3 to each logic $\mathscr{L}$. Then, given a decidable $\mathscr{L}$-theory T which has an $\mathscr{L}$-model, we will be able to show that T has a decidable $\mathscr{L}$-model, thus proving an Effective Completeness Theorem for $\mathscr{L}$.

As in Construction 3.6, we will construct a decidable model by building a sequence of FKDs. We will satisfy the completeness requirements at each existing world in stages, and create new worlds as "$\diamond$-witnessses." To satisfy the completess requirements, we would like to use the representing formula of a FKD, as in Definition 3.4.

It appears that we run into a problem here. For example, suppose that we are trying to build a **T**-model for a decidable **T**-theory. Then we need the possibility relation in the model to be reflexive. An obvious way to do this would be to make the possibility relations in the FKDs. But this would violate the tree structure of the FKDs, which is essential to defining the representing formulas. Recall that the representing formula Ψ^D of a FKD D is defined by induction on the tree structure of D, with the base case as the leaves. But there are no such leaves if the possibility relation of D is reflexive; every world has at least one successor, namely itself.

For example, take a FKD $D = (W = \{w_0\}, R = \{(w_0, w_0)\})$. Then, according to our Definition 3.4, the representing formula should be

$$\Psi^D = \Psi_0 = \bigwedge(\{\phi \mid \phi \in w_0\} \cup \{\diamond\Psi_j \mid (w_0, w_j) \in R\}) = \wedge(\{\phi \mid \phi \in w_0\} \cup \diamond\Psi_0$$

so Ψ^D is not well-defined.

We will avoid these problems by working only with *tree* FKDs as before, so that we can safely define the representing formulas. But now we will use those representing formulas to build FKDs which are consistent with respect to the class of $\mathscr{L}$-models, i.e., that have witnessing $\mathscr{L}$-models. To do so, we use a version of the Testing Lemma adapted to $\mathscr{L}$.

Lemma 5.6 (Testing lemma for $\mathscr{L}$). *Fix a modal language L. For a theory T, a tree FKD D, and a sentence ϕ (all in the language L), the following are equivalent*:

(1) $D + \{\phi \in w_i\}$ *is T-consistent with respect to the class of $\mathscr{L}$-models, i.e., there is an $\mathscr{L}$-model witnessing* $D + \{\phi \in w_i\}$;
(2) $T \not\models_{\mathscr{L}} \neg\Psi^{D+\{\phi \in w_i\}}$;
(3) $T \not\vdash_{\mathscr{L}} \neg\Psi^{D+\{\phi \in w_i\}}$;
(4) $w_i \models \neg\phi$ *is not tableau deducible from D and T using $\mathscr{L}$-rules.*

Proof. The proof is almost exactly like the proof we gave of the original Testing Lemma, i.e., for the logic **K** (see the proofs of Lemmas 3.5 and 4.4.).

(1) ⇒ (2): In the proof of Lemma 3.5, we showed that if M is a model of T which witnesses $D + \{\phi \in w_i\}$ via f, then $(M, f(w_0)) \models \Psi^{D+\{\phi \in w_i\}}$, and hence $T \not\models \neg\Psi^{D+\{\phi \in w_i\}}$. Here, the witnessing model M is an $\mathscr{L}$-model by assumption, and so we conclude that $T \not\models_{\mathscr{L}} \neg\Psi^{D+\{\phi \in w_i\}}$.

(2) ⇒ (3): By the soundness of tableau provability using $\mathscr{L}$-rules with respect to $\mathscr{L}$-models.

(3) ⇒ (4): As in the proof of Lemma 4.4, prove the contrapositive. From a contradictory tableau from D and T with root $Fw_i \models \neg\phi$ *using $\mathscr{L}$-rules*, construct a tableau proof of $\neg\Psi^{D+\{\phi \in w_i\}}$ from T, also using $\mathscr{L}$-rules.

(4) ⇒ (1): For the proof of Lemma 4.4, we showed that if $w_i \models \neg\phi$ is not tableau deducible from D and T, then a noncontradictory path through the CST from D and T with root $Fw_i \models \neg\phi$ yields a countermodel of T which witnesses $D + \{\phi \in w_i\}$. Now suppose that $w_i \models \neg\phi$ is not tableau deducible from D and T *using the $\mathscr{L}$-rules*. Then the CST using the $\mathscr{L}$-rules has a noncontradictory path, which yields a model of T witnessing $D + \{\phi \in w_i\}$. Since the CST was generated using the $\mathscr{L}$-rules, the resulting model is a $\mathscr{L}$-model (just as in the proof of the Completeness Theorem for $\mathscr{L}$; see [10]). Thus, we have a $\mathscr{L}$-model which witnesses $D + \{\phi \in w_i\}$. □

We will use this Testing Lemma to effectively construct a sequence of tree FKDs which are T-consistent with respect to the class of $\mathscr{L}$-models. From this sequence we will define a decidable $\mathscr{L}$-model for the given decidable $\mathscr{L}$-theory T, thus proving an Effective Completeness Theorem for $\mathscr{L}$.

But how do we get from *tree* FKDs to an $\mathscr{L}$-model, in which the possibility relation is reflexive/transitive/...? For instance, how does using the Testing Lemma for **K4** to define tree FKDs lead us to a Kripke model with a transitive possibility relation, even though the possibility relations of the tree FKDs are not transitive?

The simple answer is that to define the model from the tree FKDs, we will "close off" the possibility relation, so as to construct a (reflexive/transitive/...) model. Why does this work? In particular, why will the Truth Lemma still go through for this (reflexive/transitive/...) model?

One view is that although the tree FKDs do not explicitly have the properties corresponding to $\mathscr{L}$ (i.e., reflexive for **T**, transitive for **K4**, etc.), we may think that using the Testing Lemma for $\mathscr{L}$ implies that the possibility relation of the tree FKDs "implicitly" have the corresponding property. We can explain this in two different ways, semantically and syntactically, according to two clauses of the Testing Lemma:

- semantically: using clause (1) of the Testing Lemma for $\mathscr{L}$, we will define the FKDs D_n so that each D_n has a witnessing $\mathscr{L}$-model. When we look within the class of $\mathscr{L}$-models for witnessing models, the possibility relation of D_n is "converted" into a possibility relations in the witnessing model which has the special property corresponding to $\mathscr{L}$.

For example, suppose we are carrying out a construction of a decidable **K4**-model, and we have a FKD D where

$$w \rightarrow w' \rightarrow w''.$$

Since we use the Testing Lemma for **K4**, we look for a **K4**-model which witnesses D. Suppose $M = (W_M, R_M, \ldots)$ is such a **K4**-model which witnesses D via f. Then $(f(w), f(w')), (f(w'), f(w'')) \in R_M$, since $(w, w'), (w', w'') \in R_D$. Since M is a **K4**-model, R_M is transitive, so $(f(w), f(w'')) \in R_M$ as well.

Thus, the "missing" edge from w to w'' in D, which we would expect to be there if we are going to use D to construct a **K4**-model, is in any **K4**-model which witnesses D. We describe this situation by saying that using the Testing Lemma for **K4** on D means the edge is there implicitly.

- syntactically: using clause (4) of the Testing Lemma for $\mathscr{L}$, we will define the FKDs D_n so that there is no contradictory tableau from D_n using the $\mathscr{L}$-rules. But the $\mathscr{L}$-rules mean that the possibility relation R of a FKD D is again "converted" into a possibility relation with the special property corresponding to $\mathscr{L}$.

Taking the same example as above, any tableau from D using the **K4**-rules will have the entries wRw' and $w'Rw''$ on its stem. Then the **K4**-rules means wRw'' can be added to any branch of the tableau. Thus, the "missing" edge from w to w'' is put in by the tableau rules, and can be used in any attempt to derive a contradictory tableau from D.

5.3. The construction

Now that we have a Testing Lemma for $\mathscr{L}$, we can use it to construct a decidable $\mathscr{L}$-model for a given decidable $\mathscr{L}$-theory. The construction is essentially the same as Construction 3.6. We will construct a sequence of tree FKDs D_n, by satisfying completeness requirements at existing worlds and creating new worlds as $\diamond$-witnesses. But now when we attempt to test the consistency of an extended diagram $D_n + \{\phi_e \in w_i^n\}$, we will check the satisfiability of the representing formula Ψ^D with respect to the class of $\mathscr{L}$-models. This will ensure that each tree FKD D_n we build has a witnessing $\mathscr{L}$-model. This will let us define a decidable $\mathscr{L}$-model from the sequence of tree FKDs.

Construction 5.7. Given a decidable $\mathscr{L}$-theory T which has a $\mathscr{L}$-model, the construction will produce a $\mathscr{L}$-model M. $\square$

Fix a computable set of new constants C, an effective enumeration $\phi_0, \phi_1, \ldots$ of the sentences in the extended language $L \cup C$, and $\pi : \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ as in Construction 3.6.

Stage -1: $D_0 = (\{w_0^0 = \emptyset\}, R_0 = \emptyset)$.

Stage n: By induction, we have a FKD $D_n = (W_n = \{w_0^n, \ldots, w_p^n\}, R_n)$. Suppose $\pi(n) = (i, e)$. If $i > p$, simply put $D_{n+1} = D_n$, update indices from n to $n+1$, and go on to stage $n+1$.

If $i \leqslant p$, then we shall satisfy the eth completeness requirement at w_i. Let $D = D + \{\phi_e \in w_i^n\}$. Using the decidability of T, effectively check whether $T \models_{\mathscr{L}} \neg \Psi^D$. Proceed as in the analogous step of Construction 3.6.

Now we will define a Kripke model M from the sequence of FKDs D_n. The possible worlds and the interpretation of the model are defined from the D_n exactly as in Construction 3.6. The only difference is in defining the possibility relation.

Before, we just accumulated the possibility relations of the D_n. But now (for $\mathscr{L}$ other than $\mathbf{K}$), since we have only constructed *tree* FKDs, accumulating the possibility relations R_n of the FKDs will not produce a $\mathscr{L}$-model. Hence, to produce a $\mathscr{L}$-model, we "close off" the R_n so that the possibility relation of M has the desired property.

For each i such that w_i^n occurs in some D_n, let

$$w_i = \bigcup_{n=0}^{\infty} w_i^n.$$

Now define $M = (W, R_M, D, I)$ as follows:

- the set of possible worlds is $W = \{w_i : w_i^n \text{ occurs in some } D_n\}$;
- the domain D of the model is the set of new constants C;
- the interpretation I is defined from the syntactic contents of the w_i: for each n-place relation symbol P, n-tuple $\bar{c}$ of elements of C, and possible world w_i,

$$\bar{c} \in I(w_i)(P) \quad \Leftrightarrow \quad P(\bar{c}) \in w_i;$$

- to define the possibility relation R_M, first let R be the accumulation of the R_n:

$$(w_i, w_j) \in R \iff (w_i^n, w_j^n) \in R_n \text{ for some } n;$$

Now to produce an $\mathscr{L}$-model, we define R_M by "closing off" R in the appropriate way. In the case that $\mathscr{L}$ is

  - **K**, let $R_M = R$. (This is simply Construction 3.6.)
  - **T**, let R_M be the reflexive closure of R.
  - **K4**, let R_M be the transitive closure of R.
  - **S4**, let R_M be the reflexive and transitive closure of R.
  - **K5**, let R_M be the transitive and Euclidean closure of R.
  - **S5**, let R_M be the complete closure of R.

Notice that we checked whether $T \models_{\mathscr{L}} \neg\Psi^D$ within the construction. Hence, the D_n are constructed based on criteria which maintain T-consistency with respect to the class of $\mathscr{L}$-models.

Lemma 5.8 (Consistency Lemma). *Each FKD D_n is T-consistent with respect to the class of $\mathscr{L}$-models.*

Proof. The proof is very similar to the proof of the original Consistency Lemma (Lemma 3.7). Basically, we "relativize" that proof to the class of $\mathscr{L}$-models. By induction on n:

Base case: D_0 is T-consistent with respect to the class of $\mathscr{L}$-models since T has an $\mathscr{L}$-model by hypothesis.

Induction step: Assume D_n is T-consistent with respect to the class of $\mathscr{L}$-models. To show that D_{n+1} is T-consistent with respect to the class of $\mathscr{L}$-models, consider how D_n is extended to D_{n+1} at stage n:

(1) In the case that $T \models_{\mathscr{L}} \neg\Psi^D$, no $\mathscr{L}$-model of T witnesses D (by the Testing Lemma for $\mathscr{L}$). But since D_n is T-consistent with respect to to the class of $\mathscr{L}$-models, some $\mathscr{L}$-model M of T witnesses D_n via f. Then it must be that $(M, f(w_{in})) \models \neg\phi_e$, so M also witnesses D_{n+1} with respect to to the class of $\mathscr{L}$-models.

(2) In the case that $T \not\models_{\mathscr{L}} \neg\Psi^D$, there is an $\mathscr{L}$-model M of T which witnesses D. So in particular $(M, f(w_i^n)) \models \phi_e$. As before M witnesses D_{n+1} with respect to to the class of $\mathscr{L}$-models. $\square$

Using this Consistency Lemma, we prove a Closure Lemma and a Truth Lemma, as in Sections 3.6 and 3.7.

Lemma 5.9 (Closure Lemma). *The following are true for each $w_i \in W$:*

(1) *for each sentence ϕ in the extended language $L \cup C$, exactly one of ϕ or $\neg\phi$ is in w_i;*

(2) $(\phi \wedge \psi) \in w_i \Leftrightarrow \phi \in w_i$ *and* $\psi \in w_i$;

(3) $\exists x\psi(x) \in w_i \Leftrightarrow$ *there exists* $c \in C$ *such that* $\psi(c) \in w_i$;

(4) $\diamond\psi \in w_i \Leftrightarrow$ *there exists* $w_j \in W$ *such that* $(w_i, w_j) \in R_M$ *and* $\psi \in w_j$;
(5) $\Box\psi \in w_i \Leftrightarrow \psi \in w_j$ *for every* $w_j \in W$ *such that* $(w_i, w_j) \in R_M$;
(6) $T \subseteq w_i$.

Proof. The proof follows from the Consistency Lemma: for each of the clauses except (4) and (5), the proof is exactly the same as for the original Consistency Lemma (Lemma 3.7).

The forward direction of (4) is also the same as before: if $\diamond\psi \in w_i^n$, then we created a $\diamond$-witness w_j^n such that $(w_i^n, w_j^n) \in R_n$ and $\psi \in w_j^n$. Then $(w_i, w_j) \in R \subseteq R_M$ and $\psi \in w_j$.

For the backward direction of (4), we must take into account that we have defined R_M by "closing off" R. So suppose $(w_i, w_j) \in R_M$ and $\psi \in w_j$ but $\diamond\psi \notin w_i$; then $\neg\diamond\psi \in w_i$. Find n such that w_i^n and w_j^n exist in D_n, $\psi \in w_j^n$, and $\neg\diamond\psi \in w_i^n$.

Note that we cannot assume that $(w_i^n, w_j^n) \in R_n$. It may be that $(w_i, w_j) \notin R$ but $(w_i, w_j) \notin R_M$ because it is in the (reflexive/transitive/...) closure of R (in the case that $\mathscr{L} = \mathbf{T}/\mathbf{K4}/\ldots$). But it *is* true that (w_i^n, w_j^n) is in the (reflexive/...) closure of R_n. Then there cannot be *an* $\mathscr{L}$*-model* $M = (W_M, R_M, \ldots)$ which witnesses such a D_n, which contradicts the fact that each D_n is T-consistent with respect to the class of $\mathscr{L}$-models. Hence, $(w_i, w_j) \in R_M$ and $\psi \in w_j$ implies $\diamond\psi \in w_i$.

The proof of clause (5) uses the same reasoning, so we omit the details. $\Box$

The Truth Lemma follows from the Consistency Lemma, as before.

Lemma 5.10 (Truth Lemma). *For each* $w_i \in W$ *and each sentence* ϕ *of* $L \cup C$,

$$(M, w_i) \models \phi \Leftrightarrow \phi \in w_i.$$

Theorem 5.11. *Every decidable* $\mathscr{L}$*-theory in a modal language* L *has a decidable* $\mathscr{L}$*-model.*

Proof. For a given decidable $\mathscr{L}$-theory T, take the Kripke model M produced by Construction 5.7.

That truth is decidable in M (i.e., that the set $\{(w, \phi) : (M, w) \models \phi\}$ is computable) follows from the Truth Lemma and the Construction above, in the same way as in the proof of Theorem 3.11.

We show that R_M is decidable. Recall that in the Construction R is the accumulation of the R_n, which we "closed off" to define R_M. In the case that $\mathscr{L}$ is

- **K**: this was the case covered in Theorem 3.11, where we showed $R = R_M$ is decidable.
- **T**: since R_M is the reflexive closure of R, this follows easily from the fact that R is decidable; given $w_i, w_j \in W$, $(w_i, w_j) \in R_M$ iff $(w_i, w_j) \in R$ or $w_i = w_j$.
- **K4**: recall that R_M is the transitive closure of R. Given $w_i, w_j \in W$, determine whether $(w_i, w_j) \in R_M$ as follows: search for a stage n of the construction in which both w_i^n and w_j^n exist (in the FKD $D_n = (W_n, R_n)$). Then $(w_i, w_j) \in R_M$ iff w_j^n is reachable from w_i^n via R_n. The latter condition is clearly decidable since (W_n, R_n) is a finite graph.

- **S4**: since R_M is the reflexive and transitive closure of R, this follows from the previous two cases.
- **K5**: Recall that R_M is the transitive and Euclidean closure of R. We claim that for every $w_i, w_j \in W$, $(w_i, w_j) \in R_M$ unless $w_j = w_0$; then R_M is certainly decidable. Suppose $w_j \neq w_0$. If $w_i = w_0$, then clearly $(w_i, w_j) \in R_M$, since there is an R-path from w_0 to every other element of W and R_M is the transitive closure of R.

  By the same reasoning, if $w_i \neq w_0$ then $(w_0, w_i) \in R_M$ and $(w_0, w_j) \in R_M$. Then $(w_i, w_j) \in R_M$ since R_M is Euclidean.
- **S5**: since R_M be the complete closure of R, this is trivial: $(w_i, w_j) \in R_M$ for every $w_i, w_j \in W$. $\square$

6. Domain assumptions

In this section we focus on the domains of the Kripke models we construct. According to Definition 2.4, a Kripke model $M = (W, R, D, I)$ has a domain D. Each of the constructions of decidable Kripke models presented thus far started with a set of new constants C, which eventually became the domain of the model.

This is the simplest and most basic type of domain assumption for Kripke models. These models are usually called *constant domain* Kripke models. Recall that we can view a Kripke model as a collection of classical models, one associated with each possible world $w \in W$. According to our definition of a Kripke model, each of these classical models has the same domain. We will now study Kripke models in which this is not necessarily the case. Instead of being a fixed set of elements, D will now be a *domain function* which associates with each $w \in W$ a set of elements $D(w)$. We will refer to $D(w)$ as the domain of the possible world w.

Thus, in constant domain Kripke models the domain function is a constant function, which assigns the same domain to each possible world. We will consider two different types of Kripke models in which $D(w)$ can vary for different $w \in W$. In one, there is no restriction whatsoever on the domain function D; the domains may vary freely. In such models, the interpretation I must interpret the atomic relation symbols at each world w over all the domains of all the worlds.

This is a somewhat counterintuitive definition of the interpretation I. It is perhaps more natural to expect I to interpret the atomic relations at each world w only over the domain of w. For this to make sense, however, we must require that the domains be monotonically increasing in the possibility relation R.

We call these two types of models *varying domain* Kripke models and *monotonic domain* Kripke models, respectively. We will adapt our construction to produce a decidable model of each type.

6.1. Monotonic and varying domain Kripke models

We begin with the definitions monotonic and varying domain Kripke models, formalizing the discussion above and generalizing our original definition of a Kripke model (Definition 2.4).

Definition 6.1. A Kripke model is a tuple $M = (W, R, D, I)$ where W is a set of possible worlds, R is a binary possibility relation on W, and D is a domain function which assigns to each $w \in W$ a set $D(w)$. Moreover, M is a

(1) *constant domain Kripke model* if there is a set D such that $D(w) = D$ for all $w \in W$; $I(w)(P) \subseteq D^n$ for each n-place relation symbol P; and $I(c) \in D$ for each constant symbol c.
(2) *varying domain Kripke model* if $I(w)(P) \subseteq (\bigcup_{w \in W} D(w))^n$ for each n-place relation symbol P, and $I(c) \in \bigcup_{w \in W} D(w)$ for each constant symbol c.
(3) *monotonic domain Kripke model* if $I(w)(P) \subseteq D(w)^n$ for each n-place relation symbol P; $I(w)(c) \in D(w)$ for each constant symbol c; and $(w, w') \in R$ implies $D(w) \subseteq D(w')$ and $I(w)(c) = I(w')(c)$ for each constant symbol c.

As before, we will work with models that have constants which name each of the elements in the model. We described how to do this for constant domain models in Section 2.2. Similarly, for a varying domain model M we expand L to L^M by adding a constant symbol c_a for each $a \in \bigcup_{w \in W} D(w)$, and put $I(c_a) = a$ for each such a.

For monotonic domain models the situation is slightly different. For a monotonic domain model M, we obtain an expanded language $L(w)$ for each possible world w of M by adding a constant symbol c_a for each element $a \in D(w)$. Then we expand M by putting $I(w)(c_a) = a$ for each $a \in D(w)$. Note that this expansion satisfies the condition that $I(w)(c_a) = I(w')(c_a)$ in the case $(w, w') \in R$, for each $a \in D(w) \subseteq D(w')$.

We now give the definition of truth for each kind of domain assumption. The key clauses in these definitions are the quantifier clauses. They reflect the idea that, at a given possible world $w \in W$, the quantifiers quantify over $D(w)$.

Definition 6.2 (Truth in monotonic domain models). We give the conditions under which a sentence ϕ of $L(w)$ is true in a monotonic domain model $M = (W, R, D, I)$ at a possible world $w \in W$ (notation: $(M, w) \models \phi$):

(1) $(M, w) \models P(c_0, \ldots, c_n)$ iff $(I(w)(c_0), \ldots, I(w)(c_n)) \in I(w)(P)$ (where $c_0, \ldots, c_n$ are constants in $L(w)$);
(2) $(M, w) \models \phi \wedge \psi$ iff $(M, w) \models \phi$ and $(M, w) \models \psi$;
(3) $(M, w) \models \neg\phi$ iff $(M, w) \not\models \phi$;
(4) $(M, w) \models \exists x \phi(x)$ iff $(M, w) \models \phi(c)$ for some constant c in $L(w)$;
(5) $(M, w) \models \Diamond \phi$ iff $(M, w') \models \phi$ for some w' such that $(w, w') \in R$;
(6) $(M, w) \models \Box\phi$ iff $(M, w') \models \phi$ for every w' such that $(w, w') \in R$.

Note that the constants in $L(w)$ name exactly the elements of $D(w)$. Thus, the clause for $\exists x \phi(x)$ reflects the idea that the existential quantifier quantifies over $D(w)$.

For definition of truth in a varying domain model, we need to alter the clauses for the atomic sentences and for the quantifiers.

Definition 6.3 (Truth in varying domain models). We give the conditions under which a sentence of L^M is true is true in a varying domain model $M = (W, R, D, I)$ at a possible world $w \in W$:

(1) $(M,w) \models P(c_0,\ldots,c_n)$ iff $(I(c_0),\ldots,I(c_n)) \in I(w)(P)$ (where $c_0,\ldots,c_n$ are constants in L^M);
(2) same as above;
(3) same as above;
(4) $(M,w) \models \exists x\phi(x)$ iff $(M,w) \models \phi(c_a)$ for some $a \in D(w)$;
(5) same as above;
(6) same as above.

We introduce consequence relations relativized to each class of models. For a set of sentences T and a sentence ϕ in a modal language L:

- $T \models_{VD} \phi$ iff every varying domain model which is a model of T is also a model of ϕ.
- $T \models_{MD} \phi$ iff every monotonic domain model which is a model of T is also a model of ϕ.

Finally, a set of sentences T is a *varying domain* (*monotonic domain*) *theory* if it is closed with respect to $\models_{VD}$ ($\models_{MD}$).

We also generalize the notion of a decidable Kripke model to the varying domain and monotonic domain cases.

Definition 6.4. A varying domain (monotonic domain) Kripke model $M=(W,R,D,I)$ for the language L is *decidable* if the sets W, $D(w)$ (for each $w \in W$), the relation R and the relation

$$\{(w,\phi):(M,w) \models \phi\}$$

is computable, where ϕ ranges over the sentences of L^M $(L(w))$.

6.2. Varying domains

In our construction of a decidable constant domain model (Construction 3.6), we worked over a single infinite set of new constants C, which became the domain of the model. For both of the new constructions in the following sections—of a decidable varying domain model and of a decidable monotonic domain model—we will need an infinite collection of such sets $\{C_i : i \in \omega\}$, which are pairwise disjoint.

Both constructions will proceed in much the same manner as the previous ones. We will build a sequence of tree FKDs. At each stage we will satisfy a completeness requirement with respect to some sentence ϕ at some existing possible world w_i.

Moreover, C_i will be associated with the possible world w_i in both constructions. In the varying domain construction, we will put C_i as the domain of w_i. Which sentences is it necessary to decide at w_i over the course of the construction? For a varying domain model $M=(W,R,Dom,I)$, the interpretation $I(w_i)(P)$ of an n-place relation symbol P at a possible world $w_i \in W$ is a subset of $(\bigcup_{w\in W} Dom(w))^n$. Hence, we must decide atomic sentences $P(\bar{c})$ for every tuple $\bar{c}$ from $(\bigcup_{i\in\omega} C_i)^n$, and so we must decide all the sentences from $L \cup C_0 \cup C_1 \cup C_2 \cup \cdots$ at each w_i.

We will accomplish this in the construction by satisfying completeness requirements with respect to the sentences of $L \cup C_0 \cup C_1 \cup \cdots \cup C_n$ once w_n has been created. Thus, we are led to the following definition.

Definition 6.5. A *varying domain FKD* $D = (W_D = \{w_0, \ldots, w_n\}, R_D)$ is a FKD such that each $\phi \in w_i$ is a sentence of $L \cup C_0 \cup C_1 \cup \cdots \cup C_n$.

A witnessing model for a varying domain FKD D should be a varying domain model which, in addition to preserving the truth of sentences and the possibility relation of D, has elements corresponding to the constants from the C_i which occur in the FKD. Moreover, those elements must occur in the right domains in the witnessing model.

Definition 6.6. A varying domain Kripke model $M = (W_M, R_M, Dom, I)$ *witnesses* a varying domain FKD $D = (W_D, R_D)$ via f if M is a model for the language $L \cup C_0 \cup C_1 \cup \cdots \cup C_n$ and $f : W_D \to W_M$ is a function such that

- if $(w_i, w_j) \in R_D$, then $(f(w_i), f(w_j)) \in R_M$;
- for each constant symbol $c \in C_i$ which occurs in D, $I(c) \in Dom(f(w_i))$;
- if $\phi \in w_i$, then $(M, f(w_i)) \models \phi$.

As usual, we say a varying domain FKD D is *T-consistent* if there exists a varying domain model of T which witnesses D.

As before, w_i in the FKD corresponds to $f(w_i)$ in the witnessing model. Since we plan to make C_i the domain of w_i in the varying domain model we are constructing, each $c \in C_i$ that is mentioned in the given FKD should name an element that lives in $Dom(f(w_i))$. Hence, we require that $I(c) \in Dom(f(w_i))$ for such $c \in C_i$.

We can now see why we restricted the sentences in a FKD $D = (W = \{w_0, \ldots, w_n\}, R)$ to be from $L \cup C_0 \cup C_1 \cup \cdots \cup C_n$. If $i > n$, we would have no way of guaranteeing through our definition that a witnessing model has an element corresponding to $c \in C_i$ which lives in the right place. (The right place would be $Dom(f(w_i))$, but because $i > n$, w_i does not exist in D.)

Next we define a representing formula for a varying domain FKD. It will look very much like the definition above for the constant domain case (Definition 3.4). But now the representing formula must syntactically represent the notion that each $c \in C_i$ names an element in the possible world corresponding to w_i. We will use existential quantifiers to express the existence of such elements. We will also use equality. Hence, we assume that each modal language L has a two-place relation symbol "$=$" which we use to represent equality. We want to consider Kripke models for the language L in which the interpretation of $=$ is true equality. [5]

Definition 6.7. A Kripke model $M = (W, R, D, I)$ is an *equality model* if $I(w)(=)$ is the equality relation for each $w \in W$.

[5] Consult Chapter 7 of [4] for further details on equality in modal logic.

Note that this definition applies to all three types of Kripke models: constant domain, varying domain, and monotonic domain. In constant domain equality models, $I(w)(=)$ is the equality relation on D; in varying domain models, on $(\bigcup_{w\in W} D(w))$; and in monotonic domain models, on $D(w)$. We will need it only for the varying domain case.

6.3. The representing formula

Now we can define a representing formula for a varying domain tree FKD.

Definition 6.8. Suppose $D=(W=\{w_0,\dots,w_n\},R)$ is a varying domain tree FKD with root node w_0. We associate with each node $w_i\in W$ a formula Ψ_i, defined by induction.

In both clauses below, $\bar{c}_i=(c_{i0},\dots,c_{im})$ denotes the tuple of all constants from C_i occurring in D; $\bar{x}_i=(x_{i0},\dots,x_{im})$ is a corresponding tuple of fresh variables; and $\bar{x}_i=\bar{c}_i$ denotes the conjunction $x_{i0}=c_{i0}\wedge\cdots\wedge x_{im}=c_{im}$.

- If w_i is a leaf, then $\Psi_i=\exists\bar{x}_i(\bar{x}_i=\bar{c}_i)\wedge(\bigwedge\{\phi\mid\phi\in w_i\})$.
- If w_i is a not a leaf, then

$$\Psi_i=\exists\bar{x}_i(\bar{x}_i=\bar{c}_i)\wedge\left(\bigwedge\{\phi\mid\phi\in w_i\}\right)\wedge(\bigwedge\{\Diamond\Psi_j\mid(w_i,w_j)\in R\}).$$

Finally, let $\Psi^D=\Psi_0$, the formula associated with the root node.

Lemma 6.9 (Varying domain testing lemma). *For a varying domain FKD $D=(W_D,R_D)$ and a theory T, the following are equivalent*:

(1) *$D+\{\phi\in w_i\}$ is T-consistent, i.e., there exists an varying domain equality model of T witnessing $D+\{\phi\in w_i\}$*;

(2) $T\not\models_{VD}\neg\Psi^{D+\{\phi\in w_i\}}$.

Proof. The proof is very similar to the proof of the original Testing Lemma (Lemma 3.5). We simply adapt the ideas of that proof to the context of varying domains.

- (1)$\Rightarrow$(2): Suppose $M=(W_M,R_M,Dom,I)$ is a varying domain equality model of T which witnesses $D+\{\phi\in w_i\}$. As in the proof of Lemma 3.5, we prove by induction that $(M,f(w_i))\models\Psi_i$ for each $w_i\in W_D$. Then $(M,f(w_0))\models\Psi_0$, showing that $T\not\models_{VD}\neg\Psi^{D+\{\phi\in w_i\}}$.

  The only difference from the original case is that now each Ψ_i has a conjunct $\exists\bar{x}_i(\bar{x}_i=\bar{c}_i)$, where $\bar{c}$ is the tuple of all constants from C_i which occur in $D+\{\phi\in w_i\}$. Take any such $c\in C_i$ which occurs in $D+\{\phi\in w_i\}$. Since M witnesses $D+\{\phi\in w_i\}$, $I(c)\in Dom(f(w_i))$, and so $(M,f(w_i))\models\exists x(x=c)$. Therefore, $(M,f(w_i))\models\exists\bar{x}(\bar{x}=\bar{c})$ as desired.
- (2)$\Rightarrow$(1): Suppose $T\not\models_{VD}\neg\Psi^{D+\{\phi\in w_i\}}$. Then there is a varying domain (equality) model $M=(W_M,R_M,Dom,I)$ of T with $w\in W_M$ such that $(M,w)\models\Psi^{D+\{\phi\in w_i\}}$. We show that M witnesses $D+\{\phi\in w_i\}$.

Define $f: W_D \to W_M$ as in the proof of Lemma 3.5. Then

- $(w_i, w_j) \in R_D \Rightarrow (f(w_i), f(w_j)) \in R_M$;
- $\phi \in w_i \Rightarrow (M, f(w_i)) \models \phi$

by the same reasoning as we gave there.

So all that remains is to show that $I(c) \in Dom(f(w_i))$ for each constant symbol $c \in C_i$ which occurs in $D + \{\phi \in w_i\}$. By the same reasoning which shows that $(M, f(w_i)) \models \phi$ for each $\phi \in w_i$, we see that

$$(M, f(w_i)) \models \exists \bar{x}_i(\bar{x}_i = \bar{c}_i),$$

where $\bar{c}_i$ is the tuple of all constants from C_i occurring in $D + \{\phi \in w_i\}$ (because $\exists \bar{x}_i(\bar{x}_i = \bar{c}_i)$, like each $\phi \in w_i$, is a conjunct of Ψ_i). Therefore,

$$(M, f(w_i)) \models \exists x(x = c)$$

for each $c \in C_i$ which occurs in $D + \{\phi \in w_i\}$. Then there is some $a \in Dom(f(w_i))$ such that $(M, f(w_i)) \models (c_a = c)$, meaning $I(c_a) = I(c)$. Thus, $I(c) = a \in Dom(f(w_i))$ as desired. $\square$

6.4. The varying domain construction

Using the Varying Domain Testing Lemma, we can now carry out the varying domain construction.

Construction 6.10. Given a decidable varying domain theory T which has a varying domain model, the construction will produce a varying domain Kripke model M.

Begin by fixing an infinite collection $\{C_i : i \in \omega\}$ of infinite sets of constants, which are mutually disjoint and each disjoint from L. We will construct a sequence of varying domain FKDs D_n. For that purpose, fix enumerations $\phi_0^p, \phi_1^p, \phi_2^p, \ldots$ of all sentences in $L \cup C_0 \cup \cdots \cup C_p$, for each $p \in \omega$. We will construct the D_n by satisfying completeness requirements for all existing worlds with respect to the enumeration $\phi_0^p, \phi_1^p, \phi_2^p, \ldots$ (once w_p has been created). This will guarantee that we eventually decide every sentence of $L \cup C_0 \cup C_1 \cup C_2 \ldots$ at every possible world. Moreover, since we intend to put C_i as the domain of w_i in the model M, we will choose Henkin witnesses for existential sentences at w_i from C_i.

Stage -1: $D_0 = (\{w_0^0 = \emptyset\}, R_0 = \emptyset)$.

Stage n: By induction we have a varying domain tree FKD $D_n = (W_n = \{w_0^n, \ldots, w_p^n\}, R_n)$. Suppose $\pi(n) = (i, e)$.

If $i > p$, let $D_{n+1} = D_n$ and update indices from n to $n+1$.

If $i \leqslant p$, we shall satisfy the eth completeness requirement (with respect to the enumeration of sentences in $L \cup C_0 \cup \cdots \cup C_p$) at w_i. Let $D = D + \{\phi_e^p \in w_i^n\}$. Using the decidability of T, effectively check whether $T \models_{VD} \neg \Psi^D$.

(1) If $T \models_{VD} \neg\Psi^D$, we associate $\neg\phi_e^p$ with w_i: let $D_{n+1} = D_n + \{\neg\phi_e^p \in w_i^n\}$ and update indices.
(2) If $T \not\models_{VD} \neg\Psi^D$, we shall associate ϕ_e^p with w_i. We also satisfy a Henkin witness or a $\Diamond$-witness requirement for ϕ_e^p at w_i if necessary:
   - if $\phi_e^p = \exists x\psi(x)$, let $D_{n+1} = D_n + \{\phi_e^p, \psi(c_j) \in w_i^n\}$ where c_j is the least element of C_i not occurring in D. (Since we intend to define a varying domain model with C_i as the domain of w_i, we choose a Henkin witness c_j from C_i.)
   - if $\phi_e^p = \Diamond\psi$, let $D_{n+1} = (\{w_0^{n+1}, \ldots, w_p^{n+1}, w_{p+1}^{n+1}\}, R_{n+1})$ where $w_i^{n+1} = w_i^n \cup \{\phi_e^p\}$; $w_{p+1}^{n+1} = \{\psi\}$; $w_j^{n+1} = w_j^n$ for $j \neq i, p+1$; and $R_{n+1} = R_n \cup \{(w_i, w_{p+1})\}$.

   Otherwise just let $D_{n+1} = D_n + \{\phi_e^p \in w_i^n\}$ and update indices.

Now define a varying domain model $M = (W, R, D, I)$ from the D_n in the usual manner. We let $w_i = \bigcup_{n=0}^{\infty} w_i^n$, and then set

- $W = \{w_i : w_i^n \text{ occurs in some } D_n\}$.
- $(w_i, w_j) \in R \Leftrightarrow (w_i^n, w_j^n) \in R_n$ for some n.
- For the domain function D, we set up the construction with the intention of putting $D(w_i) = C_i$. Because we included equality in the language for the purposes of this construction, we must first "mod out" by the equality relation defined by the contents of the w_i.

  Define an equivalence relation $\sim$ on $\bigcup_{i=0}^{\infty} C_i$ as follows: $c \sim c'$ iff $(c = c') \in w_i$ for some $w_i \in W$. Let $[c]$ denote the equivalence class of $c \in C_i$ with respect to $\sim$. Then $D(w_i) = \{[c] : c' \in [c]$ for some $c' \in C_i\}$.[6]
- For each n-place relation symbol P, $w_i \in W$, and n-tuple $(d_1, d_2, \ldots, d_n)$ of elements of $\bigcup_{i=0}^{\infty} D(w_i)$, $(d_1, d_2, \ldots, d_n) \in I(w_i)(P)$ iff $P(c_1, \ldots, c_n) \in w_i$ for some (every) $c_1 \in d_1,\ c_2 \in d_2, \ldots, c_n \in d_n$.

That $\sim$ is an equivalence relation, and that the definition of I is well-defined, follows from the fact that each FKD D_n has a witnessing equality model, as we will show in the Consistency Lemma.

That M is in fact a varying domain model follows from the Closure Lemma below. We will show that for each $w_i \in W$ and each sentence ϕ of $L \cup C_0 \cup C_1 \cup \ldots$, either ϕ or $\neg\phi$ is in w_i. Hence, for each n-place relation symbol P and n-tuple $\bar{c} \in (\bigcup_{i=0}^{\infty} C_i)^n$, either $P(\bar{c}) \in w_i$ or $\neg P(\bar{c}) \in w_i$. Then $I(w_i)(P) \subseteq (\bigcup_{i=0}^{\infty} D(w_i))^n$, as required for a varying domain model.

Lemma 6.11 (Consistency Lemma). *Each D_n is T-consistent.*

Proof. By induction on n:

Base case: D_0 is T-consistent since T has a varying domain model by hypothesis.

Induction step: Assume $D_n = (W_n = \{w_0^n, \ldots, w_p^n\}, R_n)$ is T-consistent. To show that D_{n+1} is T-consistent, we look at how D_n is extended to D_{n+1} at stage n. Recall that $D = D_n + \{\phi_e^p \in w_i^n\}$.

[6] Note that the $D(w_i)$ are not necessarily disjoint. We may have $c \in C_i$ and $c' \in C_j$ such that $(c = c')$ was put in some (and hence every) $w \in W$. Then $[c] = [c'] \in D(w_i) \cap D(w_j)$.

(1) In the case that $T \models_{VD} \neg\Psi^D$, no varying domain model of T witnesses D (according to the Varying Domain Testing Lemma). But since D_n is T-consistent, some varying domain model M of T witnesses D_n via f. Then $(M, f(w_i^n)) \not\models \phi_e^p$ (otherwise M witnesses D). So $(M, f(w_i^n)) \models \neg\phi_e^p$, meaning M also witnesses D_{n+1} via f.
(2) In the case that $T \not\models_{VD} \neg\Psi^D$, there is a varying domain model M of T which witnesses D via f. So in particular $(M, f(w_i^n)) \models \phi_e^p$. We claim that M also witnesses D_{n+1}. If ϕ_e^p is not an existential nor a diamond sentence, this is clear.
   - if $\phi_e^p = \exists x\psi(x)$: since $(M, f(w_i^n)) \models \phi_e^p$, there is some $a \in Dom(f(w_i^n))$ such that $(M, f(w_i^n)) \models \psi(c_a)$. Then M witnesses D_{n+1} via f if we set $I(c_j) = a$. (We interpret the chosen Henkin witness c_j as an element a which witnesses the existential sentence. Note that we can set $I(c_j) = a$ since c_j does not occur in D.)
   - if $\phi_e^p = \Diamond\psi$: since $(M, f(w_i^n)) \models \phi_e^p$, there is a possible world w of M such that $(f(w_i^n, w)) \in R_M$ and $(M, w) \models \psi$. So M witnesses D_{n+1} via f if we expand f by setting $f(w_{p+1}^{n+1}) = w$. □

Lemma 6.12 (Closure Lemma). *For each $w_i \in W$,*

(1) *For each sentence ϕ of $L \cup C_0 \cup C_1 \cup C_2 \ldots$, exactly one of ϕ or $\neg\phi$ is in w_i*
(2) $(\phi \wedge \psi) \in w_i \Leftrightarrow \phi \in w_i$ *and* $\psi \in w_i$;
(3) $\exists x\psi(x) \in w_i \Leftrightarrow$ *there is a* $c \in C_i$ *such that* $\psi(c) \in w_i$;
(4) $\Diamond\psi \in w_i \Leftrightarrow$ *there is a* $w_j \in W$ *such that* $(w_i, w_j) \in R$ *and* $\psi \in w_j$;
(5) $T \subseteq w_i$.

Proof.
(1) Take any sentence ϕ of $L \cup C_0 \cup C_1 \cup C_2 \ldots$. Find n such that ϕ is a sentence of $L \cup C_0 \cup C_1 \cup C_2 \cdots \cup C_n$, w_i^n exists in D_n, $\pi(n) = (i, e)$, and $\phi = \phi_e^i$. By the construction, either $\phi_e^i \in w_i^{n+1}$ or $\neg\phi_e^i \in w_i^{n+1}$.
   As usual, it cannot be that both $\phi \in w_i$ and $\neg\phi \in w_i$, because that would contradict the consistency of some D_n.
(2) As in Lemma 3.8.
(3) ($\Rightarrow$) Suppose $\exists x\psi(x) \in w_i$. Then $\exists x\psi(x)$ is added to w_i^n for some n. Then recall that $\psi(c)$ is also added to w_i^n in the construction for some $c \in C_i$ (to fulfill the Henkin witness requirement).
   ($\Leftarrow$) Suppose $\psi(c) \in w_i$ for some $c \in C_i$ but $\exists x\psi(x) \notin w_i$. Then there is some n such that $\neg\exists x\psi(x) \in w_i^n$ and $\psi(c) \in w_i^n$. This clearly contradicts the consistency of D_n.
(4) As in Lemma 3.8.
(5) As in Lemma 3.8. □

Lemma 6.13 (Truth Lemma). *For each $w_i \in W$ and each sentence ϕ of $L \cup C_0 \cup C_1 \cup C_2 \cup \cdots$*

$$(M, w_i) \models [\phi] \quad \Leftrightarrow \quad \phi \in w_i,$$

where $[\phi]$ denotes the result of replacing in ϕ each $c \in C_i$ by $[c] \in D(w_i)$.

Proof. The same as for previous Truth Lemmas, by induction on the structure of ϕ. The key clause of the induction is for the existential quantifier, which follows from clause (3) of the Closure Lemma and from the definition of $D(w_i)$. $\square$

Once again, the construction and the Truth Lemma together show that truth in the model M is decidable. Thus, we have produced a decidable varying domain model of a given decidable varying domain theory T, establishing an effective completeness theorem for first-order modal logic over varying domain Kripke models.

Theorem 6.14. *Every decidable varying domain theory T has a decidable varying domain Kripke model.*

6.5. Monotonic domains

We now turn to the monotonic domain case. We will again work over a mutually disjoint infinite collection of new constants C_i, $i \in \omega$. In the varying domain construction, we made C_i the domain of w_i in the varying domain model. For the monotonic domain construction, the domain of w_i will *include* C_i. But to produce a monotonic domain model, it will also need to contain the domain of w_j if $(w_j, w_i) \in R$; similarly, the domain of w_j will include C_j, but will also contain the domain of w_k if $(w_k, w_j) \in R$; and so on. So the domains of the worlds in the model will be defined in terms of the C_i in such a way as to make them monotonically increasing in R. With this in mind, we give the following definition.

Definition 6.15. A *monotonic FKD* is a FKD $D = (W, R)$, with the added structure that $D(w_i)$ is a set of constant symbols for each $w_i \in W$ such that $(w_i, w_j) \in R$ implies $D(w_i) \subseteq D(w_j)$, and each $\phi \in w_i$ is a sentence of $L \cup D(w_i)$.

If we think of a monotonic FKD as a finite approximation to a monotonic domain Kripke model, the set $D(w_i)$ serves as the stand-in for the domain of the possible world w_i. This explains why we require a sentence $\phi \in w_i$ to be in the language $L \cup D(w_i)$.

As usual, we want to make sure the FKDs we construct have witnessing models. Like the varying domain case, a witnessing model should, in addition to preserving the truth of the sentences and the accessibility relation of the FKD, have elements corresponding to the constants occurring in the FKD which belong to the right domains of the witnessing model. This is where we will make use of the languages $L(w)$, which have constants for every element in the domain of a possible world w.

Definition 6.16. A monotonic domain Kripke model $M = (W_M, R_M, Dom, I)$ *witnesses* a monotonic FKD $D = (W_D, R_D)$ via f, g if

- $f : W_D \to W_M$ is a function such that

$$(w_i, w_j) \in R_D \Rightarrow (f(w_i), f(w_j)) \in R_M.$$

- g maps each constant symbol $c \in D(w_i)$ to a constant $g(c) \in L(f(w_i))$;

- if $\phi \in w_i$, then $(M, f(w_i)) \models \phi(g(\bar{c})/\bar{c})$, where $\bar{c}$ is the tuple of all constant symbols from $D(w_i)$ occurring in ϕ

As usual, we say a monotonic FKD D is (T-)*consistent* if there exists a monotonic domain Kripke model (of T) which witnesses D.

The function g guarantees that for each constant in $D(w_i)$ there is an element in the right domain in the witnessing model. If a constant $c \in D(w_i)$ occurs in a sentence $\phi \in w_i$, there should be an element in the domain of $f(w_i)$ corresponding to c. Such an element is named by a constant in $L(f(w_i))$, and is specified as $g(c)$. Then each sentence $\phi \in w_i$ must be true at $f(w_i)$ in the witnessing model M, but with the constants from $D(w_i)$ replaced by their counterparts from $L(f(w_i))$.

The function g was unnecessary in the previous definitions of a witnessing model, for constant domains and varying domains (Definitions 3.2 and 6.6, respectively), because in those cases it sufficed to require that a witnessing model be a model of a language containing all the constants occurring in the FKD ($L \cup C$ and $L \cup C_0 \cup \cdots \cup C_n$, respectively). The constants occurring in the FKD were simply interpreted as elements of the witnessing model. Clearly, we could recast those definitions in the manner of the one above, with each constant c in a constant domain or varying domain FKD corresponding to a constant $g(c)$ in L^M in a witnessing constant domain or varying domain model M.

6.6. The representing formula

We want a representing formula which will allow us to test whether a given monotonic FKD is consistent. This representing formula must encode that the constants of $D(w_i)$ correspond to elements in the domain of $f(w_i)$. As in the varying domain case, we will do this by existentially quantifying out the constants of the $D(w_i)$ within a nested diamond formula. The witnesses for these quantifiers will allow us to define the function g on the constants of the FKD in the proof of the new monotonic Testing Lemma.

We need to take care to existentially quantify out the constants of $D(w_i)$ at the right level in the nested diamond structure of the representing formula, so that the existential quantifiers speak about the existence of elements in the right place. We will see this at work in the corresponding Testing Lemma.

Definition 6.17. Suppose $D = (W_D = \{w_0, \ldots, w_n\}, R_D)$ is a monotonic tree FKD with root node w_0. Since D is a tree FKD, each $w_i \in W_D$ has a unique R_D-predecessor; call it w_i'. Note that by monotonicity, $D(w_i') \subseteq D(w_i)$. Let B_i denote the set $D(w_i) - D(w_i')$.

Now we will define a formula Ψ_i for each $w_i \in W$. We will quantify out the constants from B_i at the level in the nested diamond structure corresponding to w_i.[7] In both clauses below, $\bar{b}_i = (b_0, \ldots, b_m)$ denotes the tuple of all constants from B_i occurring in

[7] Note that we do not want to quantify out *all* the constants from $D(w_i)$ at that level, because $D(w_i)$ includes some constants—namely the constants from $D(w_i')$—that should be quantified out "higher" levels. That is our reason for defining the B_i.

D; $\bar{x}_i = (x_0, \dots, x_m)$ is a corresponding tuple of fresh variables; and $\bar{x}_i = \bar{b}_i$ denotes the conjunction $x_0 = b_0 \wedge \cdots \wedge x_m = b_m$.

- If w_i is a leaf, then $\Psi_i = \exists \bar{x}_i((\bigwedge\{\phi \mid \phi \in w_i\})(\bar{x}_i/\bar{b}_i))$.
- If w_i is a not a leaf, then

$$\Psi_i = \exists \bar{x}_i \left(\left(\bigwedge(\{\phi \mid \phi \in w_i\} \cup \{\Diamond \Psi_j \mid (w_i, w_j) \in R\}) \right) (\bar{x}_i/\bar{b}_i) \right).$$

Finally, let $\Psi^D = \Psi_0$, the formula associated with the root node. We call Ψ^D the *representing formula* of D.

Lemma 6.18 (Monotonic domain testing lemma). *For a monotonic* FKD $D = (W_D, R_D)$, *a theory* T, *and a sentence* ϕ, *the following are equivalent*:

(1) $D + \{\phi \in w_i\}$ *is* T*-consistent*, *i.e.*, *there exists a monotonic domain model of* T *witnessing* $D + \{\phi \in w_i\}$.
(2) $T \not\models_{MD} \neg \Psi^{D+\{\phi \in w_i\}}$.

Proof. The proof is similar to the proofs of the earlier Testing Lemmas. For brevity, we let Θ_i denote the conjunction

$$\bigwedge(\{\phi \mid \phi \in w_i\} \cup \{\Diamond \Psi_j \mid (w_i, w_j) \in R_D\})$$

for each $w_i \in W_D$ in a given monotonic FKD.

- $(1) \Rightarrow (2)$: Suppose M is a monotonic domain model of T which witnesses $D' = D + \{\phi \in w_i\}$ via f, g. Let $\bar{c}$ be the constants from the $D(w_i)$ which occur in D'. We will prove by induction that for each $w_i \in W_D$, $(M, f(w_i)) \models \Psi_i(g(\bar{c})/\bar{c})$. Then, because Ψ_0 is free of such constants, we will have showed that $(M, f(w_0)) \models \Psi^{D+\{\phi \in w_i\}}$.

  For the base case, suppose w_i is a leaf. Since M witnesses D via f, g, $(M, f(w_i)) \models \phi(g(\bar{c})/\bar{c})$ for each $\phi \in w_i$. Hence,

  $$(M, f(w_i)) \models \left(\bigwedge\{\phi \mid \phi \in w_i\} \right) (g(\bar{c})/\bar{c}).$$

  Let $\bar{b}_i$ be the elements in $\bar{c}$ which belong to B_i. Since $B_i \subseteq D(w_i)$, $g(\bar{b}_i) \subseteq L(f(w_i))$. Therefore,

  $$(M, f(w_i)) \models \exists \bar{x} \left(\left(\bigwedge\{\phi \mid \phi \in w_i\} \right) (g(\bar{c})/\bar{c}) (\bar{x}/g(\bar{b}_i)) \right)$$

  which implies

  $$(M, f(w_i)) \models \exists \bar{x} \left(\left(\bigwedge\{\phi \mid \phi \in w_i\} \right) (\bar{x}/\bar{b}_i) \right) (g(\bar{c})/\bar{c})$$

  i.e., $(M, f(w_i)) \models \Psi_i(g(\bar{c})/\bar{c})$ as desired.

  For the inductive step, suppose w_i is not a leaf. Consider each $w_j \in W_D$ such that $(w_i, w_j) \in R_D$. By hypothesis, $(f(w_i), f(w_j)) \in R_M$. By induction, $(M, f(w_j)) \models$

$\Psi_j(g(\bar{c})/\bar{c})$. Moreover, as above, $(M, f(w_i)) \models \phi(g(\bar{c})/\bar{c})$ for each $\phi \in w_i$. Hence, $(M, f(w_i)) \models \Theta_i(g(\bar{c})/\bar{c})$. Again let $\bar{b}_i$ be the elements in $\bar{c}$ which belong to B_i. Then, as above,

$$(M, f(w_i)) \models \exists \bar{x}(\Theta_i(g(\bar{c})/\bar{c})(\bar{x}/g(\bar{b}_i)))$$

which implies

$$(M, f(w_i)) \models \exists \bar{x}(\Theta_i(\bar{x}/\bar{b}_i))(g(\bar{c})/\bar{c})$$

i.e., $(M, f(w_i)) \models \Psi_i(g(\bar{c})/\bar{c})$.

- $(2) \Rightarrow (1)$: Supposing $T \not\models_{MD} \neg\Psi^{D+\{\phi\in w_i\}}$, there is a monotonic domain model $M = (W_M, R_M, Dom, I)$ of T with $w \in W_M$ such that $(M, w) \models \Psi^{D+\{\phi\in w_i\}}$. We will show that M witnesses $D' = D + \{\phi \in w_i\}$. The proof is similar to the earlier proofs of the Testing Lemmas, in that we will define f by induction, using the nested diamond structure of $\Psi^{D+\{\phi\in w_i\}}$. Here we must also define g as we go; we will use the witnesses for the existential quantifiers in $\Psi^{D+\{\phi\in w_i\}}$ to do so.

  For the base case of the root w_0, $f(w_0) = w$. Recall that $\Psi^{D+\{\phi\in w_i\}}$ is of the form

$$\exists \bar{x}(\Theta_0(\bar{x}/\bar{b}_0)),$$

where $\bar{b}_0$ is all constants from $B_0(= D(w_0))$ occurring in D'. Therefore, since $(M, f(w_0)) \models \Psi^{D+\{\phi\in w_i\}}$, there exists $\bar{c}_0 \in L(f(w_0))$ such that

$$(M, f(w_0)) \models (\Theta_0(\bar{x}/\bar{b}_0))(\bar{c}_0/\bar{x})$$

i.e., $(M, f(w_0)) \models \Theta_0(\bar{c}_0/\bar{b}_0)$. Set $g(\bar{b}_0) = \bar{c}_0$. Note that Θ_0 is a conjunction that includes among its conjuncts each $\phi \in w_0$. Hence, $(M, f(w_0)) \models \phi(g(\bar{b}_0)/\bar{b}_0)$ for each $\phi \in w_0$, where $\bar{b}_0$ includes all the constant symbols from $D(w_0)$ occurring in ϕ.

For the induction step, suppose $w_i \in W_D$ is a node other than the root. Let $w_0 \to w_{i1} \to \cdots \to w_{im} \to w_i$ be the unique path through the tree $D = (W_D, R_D)$ from the root w_0 to w_i. By induction we have defined $f(w_0), f(w_{i1}), \ldots, f(w_{im})$ and $g(\bar{b}_0) \in L(f(w_0))$, $g(\bar{b}_{i1}) \in L(f(w_{i1})), \ldots, g(\bar{b}_{im}) \in L(f(w_{im}))$ such that

$$(M, f(w_{im})) \models \Theta_{im}(g(\bar{b}_0)/\bar{b}_0)(g(\bar{b}_{i1})/\bar{b}_{i1}) \ldots (g(\bar{b}_{im})/\bar{b}_{im}),$$

where $\bar{b}_0, \bar{b}_{i1}, \ldots, \bar{b}_{im}$ are, respectively, the constants from $B_0, B_{i1}, \ldots, B_{im}$ occurring in D'.

Since $(w_{im}, w_i) \in R_D$, $\Diamond\Psi_i$ is among the conjuncts of Θ_{im}. Hence,

$$(M, f(w_{im})) \models \Diamond\Psi_i(g(\bar{b}_0)/\bar{b}_0)(g(\bar{b}_{i1})/\bar{b}_{i1}) \ldots (g(\bar{b}_{im})/\bar{b}_{im}).$$

Therefore, there exists $w \in W_M$ such that $(f(w_{im}), w) \in R_M$ and

$$(M, w) \models \Psi_i(g(\bar{b}_0)/\bar{b}_0)(g(\bar{b}_{i1})/\bar{b}_{i1}) \ldots (g(\bar{b}_{im})/\bar{b}_{im}).$$

Fix such a w and set $f(w_i) = w$. Now we will define g on $\bar{b}_i$, the constants from B_i occurring in D'. Recall that Ψ_i is of the form $\exists\bar{x}(\Theta_i(\bar{x}/\bar{b}_i))$. Hence, there exist constants $\bar{c}_i \in L(f(w_i))$ such that

$$(M, f(w_i)) \models \Theta_i(g(\bar{b}_0)/\bar{b}_0)(g(\bar{b}_{i1})/\bar{b}_{i1})\ldots(g(\bar{b}_{im})/\bar{b}_{im})(\bar{x}/\bar{b}_i)(\bar{c}_i/\bar{x})$$

i.e.,

$$(M, f(w_i)) \models \Theta_i(g(\bar{b}_0)/\bar{b}_0)(g(\bar{b}_{i1})/\bar{b}_{i1})\ldots(g(\bar{b}_{im})/\bar{b}_{im})(\bar{c}_i/\bar{b}_i).$$

Set $g(\bar{b}_i) = \bar{c}_i$ to establish the desired inductive hypothesis.

Then, since each $\phi \in w_i$ is among the conjuncts of Θ_i,

$$(M, f(w_i)) \models \phi(g(\bar{b}_0)/\bar{b}_0)(g(\bar{b}_{i1})/\bar{b}_{i1})\ldots(g(\bar{b}_{im})/\bar{b}_{im})(g(\bar{b}_i)/\bar{b}_i).$$

Notice that since $\phi \in w_i$ is a sentence of $L(D(w_i))$ and $D(w_i) = B_0 \cup B_{i1} \cup \cdots B_{im}$, $\bar{b}_0, \bar{b}_{i1}, \ldots \bar{b}_{im}, \bar{b}_i$ includes all the constants which appear in such $\phi \in w_i$. Hence, for each $\phi \in w_i$,

$$(M, f(w_i)) \models \phi(g(\bar{c})/\bar{c}),$$

where $\bar{c}$ is the tuple of all constants appearing in ϕ, as required to show that M witnesses D via f, g. $\square$

Using this Testing Lemma, we can carry out the construction of a decidable monotonic domain Kripke model.

6.7. The monotonic domain construction

Construction 6.19. Given a decidable monotonic domain theory T, the construction will produce a monotonic domain model M. We will construct a sequence of monotonic FKDs D_n. When we create an existing world w_i^n in a FKD D_n, we will specify a set of constants $D(w_i)$. We will set $D_n(w_i^n) = D(w_i)$ for each FKD D_n, and use $D(w_i)$ as the domain of w_i in the model. Hence, we will satisfy completeness requirements at w_i with respect to the language $L \cup D(w_i)$, and choose Henkin witnesses for existential sentences at w_i from $D(w_i)$. The $D(w_i)$ will be defined in terms of the sets of constants C_i.

Stage -1: $D_0 = (\{w_0^0 = \emptyset\}, R_0 = \emptyset)$. Set $D(w_0) = C_0$, and fix an enumeration $\phi_0^0, \phi_1^0, \phi_2^0, \ldots$ of all sentences in $L \cup D(w_0)$.

Stage n: By induction we have a monotonic domain FKD $D_n = (W_n = \{w_0^n, \ldots, w_p^n\}, R_n)$. We have also defined $D(w_j)$ and fixed an enumeration $\phi_0^j, \phi_1^j, \phi_2^j, \ldots$ of all sentences in $L \cup D(w_j)$, for each $j = 0, \ldots, p$. As we noted above, $D_n(w_j^n) = D(w_j)$, so (by the definition of a monotonic FKD) $(w_i^n, w_j^n) \in R_n$ implies $D(w_i) \subseteq D(w_j)$, and each $\phi \in w_j^n$ is a sentence of $L \cup D(w_j)$.

Suppose $\pi(n) = (i, e)$. If $i > p$, let $D_{n+1} = D_n$ and update indices.

If $i \leqslant p$, we satisfy the eth completeness requirement (with respect to $L \cup D(w_i)$) at w_i. Let $D = D_n + \{\phi_e^i \in w_i^n\}$. Note that ϕ_e^i is a sentence of $L \cup D(w_i)$. Using the decidability of T, effectively check whether $T \models_{MD} \neg\Psi^D$.

(1) If so, associate $\neg\phi_e^i$ with w_i: let $D_{n+1} = D_n + \{\neg\phi_e^i \in w_i^n\}$ and update indices.

(2) If not, we associate ϕ_e^i with w_i. We also satisfy a Henkin witness or a $\Diamond$-witness requirement for ϕ_e^i at w_i if necessary:

- if $\phi_e^i = \exists x\psi(x)$, let $D_{n+1} = D_n + \{\phi_e^i, \psi(c_j) \in w_i^n\}$ where c_j is the least element of $D(w_i)$ not occurring in D_n; and update indices. (Since we intend to define a monotonic domain model with $D(w_i)$ as the domain of w_i, we choose a Henkin witness from $D(w_i)$.)
- if $\phi_e^i = \Diamond\psi$, let $D_{n+1} = (\{w_0^{n+1}, \ldots, w_p^{n+1}, w_{p+1}^{n+1}\}, R_{n+1})$ where $w_i^{n+1} = w_i^n \cup \{\phi_e^i\}$; $w_{p+1}^{n+1} = \{\psi\}$; $w_j^{n+1} = w_j^n$ for $j \neq i, p+1$; and $R_{n+1} = R_n \cup \{(w_i, w_{p+1})\}$. Also, set $D(w_{p+1}) = D(w_i) \cup C_{p+1}$, and fix an enumeration $\phi_0^{p+1}, \phi_1^{p+1}, \phi_2^{p+1}, \ldots$ of all sentences in $L \cup D(w_{p+1})$. (Note that our definition of $D(w_{p+1})$ clearly satisfies the monotonicity requirement. In fact, for each w_i, $D(w_i) = C_{i_0} \cup C_{i_1} \cup \cdots \cup C_{i_m}$, where $i_0 = 0$, $i_m = i$, and $w_0 \to w_{i_1} \to w_{i_2} \to \cdots \to w_{i_m}$ is the unique path through (W_n, R_n) from the root w_0 to w_i.)

Otherwise simply let $D_{n+1} = (\{w_0^{n+1}, \ldots, w_p^{n+1}\}, R_{n+1})$ with $w_i^{n+1} = w_i^n \cup \{\phi_e^i\}$; $w_j^{n+1} = w_j^n$ for $j \neq i$; and $R_{n+1} = R_n$.

Now define a monotonic domain model $M = (W, R, D, I)$: let $w_i = \bigcup_{n=1}^{\infty} w_i^n$, and

- $W = \{w_i : w_i^n \text{ occurs in some } D_n\}$;
- $(w_i, w_j) \in R \Leftrightarrow (w_i^n, w_j^n) \in R_n$ for some n;
- $D(w_i)$ as defined over the course of the construction;
- for each n-place relation symbol P, $w_i \in W$, and n-tuple $\bar{c}$ of elements of $D(w_i)$

$$\bar{c} \in I(w_i)(P) \quad \Leftrightarrow \quad P(\bar{c}) \in w_i.$$

Note that M is a monotonic domain model since the $D(w_i)$ were defined in such a way as to be monontonically increasing in R. Also, $I(w_i)(P) \subseteq D(w_i)^n$ for each $w_i \in W$ and n-place relation symbol P, since we decide the sentences of $L \cup D(w_i)$ at w_i over the course of the construction.

The D_n are constructed based on criteria which maintain T-consistency (with respect to monotonic domain models).

Lemma 6.20 (Consistency Lemma). *Each FKD D_n is T-consistent.*

Proof. The proof is similar to the proofs of the prior Consistency Lemmas. By induction on n:

Base case: D_0 is T-consistent since T has a monotonic domain model by hypothesis.

Induction step: Assume D_n is T-consistent. To show that D_{n+1} is T-consistent, look at how D_n is extended to D_{n+1} at stage n. Recall that $D = D_n + \{\phi_e^i \in w_i^n\}$. Let $\bar{c}$ be the tuple of all constant symbols in $D(w_i)$ occurring in ϕ_e^i.

(1) In the case that $T \models \neg\Psi^D$, no monotonic domain model of T witnesses D (by the Testing Lemma). But since D_n is T-consistent, some monotonic domain model M

of T witnesses D_n via f, g. Then $(M, f(w_i^n)) \not\models \phi_e^i(\overline{g(c)}/\bar{c})$ (otherwise M witnesses D). So it must be that $(M, f(w_i^n)) \models \neg\phi_e^i(\overline{g(c)}/\bar{c})$, so M also witnesses D_{n+1}.

(2) In the case that $T \not\models \neg\Psi^D$, there is a monotonic domain model M of T which witnesses D. So in particular $(M, f(w_i^n)) \models \phi_e^i(\overline{g(c)}/\bar{c})$. We claim that M also witnesses D_{n+1}. If ϕ_e^i is not an existential nor a diamond sentence, this is clear.

- if $\phi_e^i = \exists x\psi(x)$: since $(M, f(w_i^n)) \models \phi_e^i(\overline{g(c)}/\bar{c})$, there is some $d \in L(f(w_i^n))$ such that $(M, f(w_i^n)) \models \psi(d)(\overline{g(c)}/\bar{c})$. Then M witnesses D_{n+1} via f, g if we set $g(c_j) = d$ (i.e., we take the chosen Henkin witness c_j and map it via g to an element (named by d) in the model M which witnesses the existential sentence).
- if $\phi_e^i = \Diamond\psi$: since $(M, f(w_i^n)) \models \phi_e^i(\overline{g(c)}/\bar{c})$, there is a possible world w of M such that $(f(w_i^n, w)) \in R_M$ and $(M, w) \models \psi(\overline{g(c)}/\bar{c})$. So M witnesses D_{n+1} via f, g if we expand f by setting $f(w_{p+1}^{n+1}) = w$. □

Lemma 6.21 (Closure Lemma). *For each $w_i \in W$,*

(1) *For each sentence ϕ of $L \cup D(w_i)$, exactly one of ϕ or $\neg\phi$ is in w_i.*
(2) $(\phi \wedge \psi) \in w_i \Leftrightarrow \phi \in w_i$ *and* $\psi \in w_i$.
(3) $\exists x\psi(x) \in w_i \Leftrightarrow$ *there is a* $c \in D(w_i)$ *such that* $\psi(d) \in w_i$.
(4) $\Diamond\psi \in w_i \Leftrightarrow$ *there is a* $w_j \in W$ *such that* $(w_i, w_j) \in R$ *and* $\psi \in w_j$.
(5) $T \subseteq w_i$.

Proof.
(1) Find a stage n such that w_i^n exists in D_n, $\pi(n) = (i, e)$, and $\phi = \phi_e^i$. By the construction, either $\phi_e^i \in w_i^{n+1}$ or $\neg\phi_e^i \in w_i^{n+1}$.

If both $\phi \in w_i$ and $\neg\phi \in w_i$, then they are both in some w_i^n. Let $\bar{c}$ be the tuple of all elements in $D(w_i)$ occurring in ϕ. Clearly there is no model M with a world $f(w_i^n)$ and constants $g(\bar{c})$ in $L(f(w_i^n))$ such that $(M, f(w_i^n)) \models \phi(\overline{g(c)}/\bar{c})$ and $(M, f(w_i^n)) \models \neg\phi(\overline{g(c)}/\bar{c})$. Thus, the consistency of D_n is contradicted.

(2) Same as for previous Closure Lemmas.
(3) $(\Rightarrow)$ Suppose $\exists x\psi(x) \in w_i$. Then $\exists x\psi(x)$ is added to w_i^n at some stage n. Then $\psi(c)$ is also added to w_i^n at stage n for some $c \in D(w_i)$ (to fulfill the Henkin witness requirement).

$(\Leftarrow)$ Suppose $\psi(c) \in w_i$ for some $d \in D(w_i)$ but $\exists x\psi(x) \notin w_i$. Then there is some n such that $\neg\exists x\psi(x) \in w_i^n$ and $\psi(c) \in w_i^n$. Let $\bar{c}$ be the tuple of all elements in $D(w_i)$ occurring in ϕ. Clearly there is no model M with a world $f(w_i^n)$ and constants $g(\bar{c})$, $g(d) \in L(f(w_i^n))$ such that $(M, f(w_i^n)) \models \neg\exists x\psi(x)(\overline{g(c)}/\bar{c})$ and $(M, f(w_i^n)) \models \psi(d)(\overline{g(c)}/\bar{c})$. Thus, the consistency of D_n is contradicted.

(4) Same as for previous Closure Lemmas.
(5) Same as for previous Closure Lemmas. □

Lemma 6.22 (Truth Lemma). *For each $w_i \in W$ and each sentence ϕ of $L \cup D(w_i)$*

$$(M, w_i) \models \phi \quad \Leftrightarrow \quad \phi \in w_i.$$

Proof. By induction on the structure of ϕ, as for previous Truth Lemmas. □

As before, the Truth Lemma and the Construction together establish that truth in the model M is decidable. It is clear that $D(w_i)$ is a computable set for each $w_i \in W$: as we noted in the Construction, $D(w_i) = C_{i_0} \cup C_{i_1} \cup \cdots \cup C_{i_m}$, where $i_0 = 0$, $i_m = i$, and $w_0 \to w_{i_1} \to w_{i_2} \to \cdots \to w_{i_m}$ is the unique path through (W, R) from the root w_0 to w_i. We can effectively find $C_{i_0}, C_{i_1}, \ldots, C_{i_m}$ by looking at any D_n in which w_i^n exists. Since $C_{i_0}, C_{i_1}, \ldots, C_{i_m}$ are computable sets by assumption, $D(w_i)$ is also computable.

Thus, we have produced a decidable monotonic domain model of a given decidable monotonic domain theory T, as desired.

Theorem 6.23. *Every decidable monotonic domain theory T has a decidable monotonic domain Kripke model.*

6.8. Conclusion

In the preceding sections we have proved effective completeness theorems for varying domain and monotonic domain first-order modal logic, in a similar manner as for constant domain **K** in Section 3, and for constant domain **T**, **K4**, **K5**, **S4**, and **S5** in Section 5. We stated and proved a Testing Lemma which applies to finite Kripke diagrams, and then used the Testing Lemma to construct a decidable Kripke model for a decidable theory via such FKDs.

To be precise, we have proved effective completeness theorems for the varying domain and monotonic domain versions of **K**, since no conditions were placed on the possibility relations of the Kripke models. Moreover, we stated the Varying Domain and Monotonic Domain Testing Lemmas (Lemmas 6.9 and 6.18, respectively) with only semantic clauses. This suffices to carry out the constructions (as in Section 3). We have not supplyied the corresponding syntactic clauses, as we did in the Testing Lemmas for the constant domain logics in Sections 2 and 3 (Lemmas 4.4 and 5.6). We briefly indicate how to fill in these gaps.

With regard to the Testing Lemmas, we simply note that that the Varying Domain and Monotonic Domain Testing Lemmas can be extended to include similar syntactic clauses. By using the appropriate tableau proof theory for the varying domain and monotonic domain logics. We refer the reader to [4] for the definitions of the varying domain tableau, and to [10] for the monotonic domain case. Because we use equality in the definition of the representing formula in the varying domain case, we include the tableau rules for equality; refer to [4] for these also.

Given the definitions of tableau proofs corresponding to each type of domain assumption, we can define the notion of a tableau deduction from a varying domain or monotonic domain FKD, following Definition 4.3. Then we can extend the Varying Domain and Monotonic Domain Testing Lemmas with syntactic clauses, similar to the syntactic clauses of Lemmas 4.4 and 5.6, but referring to varying domain and monotonic domain tableau.

We have chosen to omit the technical details, but we wish to emphasize that the syntactic viewpoint of tableau deductions is very useful for understanding the Testing Lemma and its role in our effective constructions of Kripke models. In particular, we can test a FKD $D + \{\phi \in w_i\}$ for T-consistency by checking whether there is a tableau deduction of $w_i \models \neg\phi$ from D and T. If there is no such tableau deduction, then the complete systematic tableau (CST) has a noncontradictory path which yields a witnessing model for $D + \{\phi \in w_i\}$. The key point is that using the appropriate set of tableau rules for the given logic to generate the CST implies that the model defined from the noncontradictory path is a Kripke model for that particular logic. Using monotonic domain tableau, as in the Monotonic Domain Testing Lemma, produces a monotonic domain Kripke model; similarly for varying domains. Note that we applied this idea in 5, with respect to the logics **T**, **K4**, **K5**, **S4**, and **S5**: using the $\mathscr{L}$-rules to generate the CST produced to an $\mathscr{L}$-model.

In fact, it should be clear that by combining the techniques used in Section 5 (for the logics **T**, **K4**, **K5**, **S4**, and **S5**) with the tools developed in this Section (for varying domains and monotonic domains), we can prove effective completeness theorems for the varying domain and monotonic domain versions of **T**, **K4**, **K5**, **S4**, and **S5** as well. Namely, we can prove Varying Domain and Monotonic Domain Testing Lemmas for each of these logics, using the Kripke models and tableau proofs corresponding to each. Then that Testing Lemma can be used to carry out a construction of a decidable varying domain or monotonic domain $\mathscr{L}$-model (of a given decidable varying domain or monotonic domain $\mathscr{L}$-theory). The technical details are straightforward, so we omit them.

In conclusion, we have established Testing Lemmas and effective completeness theorems for 18 different first-order modal logics: each combination of constant domain, varying domain, or monotonic domain; and **K**, **T**, **K4**, **K5**, **S4**, or **S5**.

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